Homework #7.

Plan for next week: Free groups and presentations of groups by generators and relators (§ 6.3).

Problems, to be submitted by Thursday, October, 27th

1. Let G be a group.

(a) Prove that if N is a normal subgroup of G, then for any $k \in \mathbb{N}$ we have $\gamma_k(G/N) = (\gamma_k G \cdot N)/N$.

(b) Now assume that G is nilpotent of class $c \geq 1$. Prove that $G/Z(G)$ is nilpotent of class (exactly) $c - 1$.

2. (a) Let R be an associative ring with 1, and let $a, b \in R$ be such that $1 + a$ and $1 + b$ are invertible. Prove the following formula

$$(1+a)^{-1}(1+b)^{-1}(1+a)(1+b) = 1 + (1+a)^{-1}(1+b)^{-1}(ab - ba).$$

(b) Let R be an associative ring with 1 and $n \geq 2$ be an integer, and let $U_n(R)$ be the upper unitriangular subgroup of $GL_n(R)$. Prove that $U_n(R)$ is nilpotent of class $n-1$ (we briefly outlined the proof in class). Note that you will need to apply (a) not to R itself but to the ring of $n \times n$ matrices over R.

3. (a) Let $n \in \mathbb{N}$ be an integer, and suppose that for every non-prime divisor m of n there are no simple groups of order m. Prove that any group of order n is solvable.

(b) Prove that any group of order $p^k q$, where $p > q$ are distinct primes, is solvable.

4. Problems 31 and 32 on page 200 of DF. **Note:** Problem 31 follows very easily from Lemma 11.2 (about the structure of minimal normal subgroups).

5. (a) Let A and B be finitely generated groups. Prove that the wreath product $A \operatorname{wr} B$ is also finitely generated. **Hint:** Recall that $A \operatorname{wr} B = C \rtimes B$ where $C = \oplus_{b \in B} A_b$ (with each $A_b \cong A$). Let S be a generating set for A, T a generating set for B, fix $b \in B$, and let S_b be the image of S under an isomorphism $A \to A_b$. Prove that $S_b \cup T$ generates $A \operatorname{wr} B$.

(b) Use (a) to give a simple example showing that a subgroup of a finitely generated group may not be finitely generated.

6. (a) Prove that a Sylow p-subgroup of S_{p^2} is isomorphic to $\mathbb{Z}_p \operatorname{wr} \mathbb{Z}_p$.

(b) (bonus) Realize the Sylow p-subgroup of S_{p^3} in terms of wreath products and prove your answer (you may skip some technical details).