Homework #6, due on Thursday, October 18th.

Plan for the next week: Nilpotent and solvable groups (the best approximation in Dummit and Foote is 6.1, but we will not follow it very closely).

1. (a) Classify all abelian groups of order $360 = 2^3 \cdot 3^2 \cdot 5$ up to isomorphism. For each isomorphism type, state the corresponding elementary divisors form and invariant factors form.

(b) Let $n \in \mathbb{N}$, and decompose n as a product of primes: $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. Find the number of non-isomorphic abelian groups of order n. Express your answer in terms of the partition function.

2. Let G be a finite abelian group. Prove that G is cyclic if and only if G does not contain a subgroup isomorphic to $B \oplus B$ for some non-trivial group B.

3. Given a finite group G and a positive integer n, denote by $a_n(G)$ the number of elements of G of order n and by $b_n(G)$ the number of elements of G of order dividing n. The goal of this problem is to prove the following theorem:

Theorem A: If G and H are finite abelian groups and $a_n(G) = a_n(H)$ for all n, then G is isomorphic to H.

(a) Let G and H be finite groups. Prove that $a_n(G) = a_n(H)$ for all n $\iff$ $b_n(G) = b_n(H)$ for all n.

(b) Suppose that $G = X \times Y$. Prove that $b_n(G) = b_n(X)b_n(Y)$.

(c) Suppose that G and H are finite abelian groups s.t. $a_n(G) = a_n(H)$ for all n. Prove that there exists a non-trivial group C s.t. $G \cong A \times C$ and $H \cong B \times C$ for some groups A and B. **Hint:** Use the classification theorem in invariant factors form.

(d) Now use (a),(b) and (c) and induction to prove Theorem A.

4. Let G be an abelian group (not necessarily finitely generated), and let $Tor(G)$ be the set of elements of finite order in G. Recall that $Tor(G)$ is a subgroup of G (since G is abelian), called the torsion subgroup of G.

(a) Prove that the quotient group $G/Tor(G)$ is torsion-free.

(b) For each prime p let $Tor_p(G)$ be the set of elements of order p^k (with $k \geq 0$) in G. Prove that each $Tor_p(G)$ is a subgroup of $Tor(G)$ and that $Tor(G)$ is a direct sum of these subgroups where p runs over all primes.

5. Let Ω be a countable set (for simplicity you may assume that $\Omega = \mathbb{Z}$, the integers). Let $S(\Omega)$ be the group of all permutations of Ω. A permutation $\sigma \in S(\Omega)$ is called *finitary* if it moves only a finite number of points, that is, the set $\{i \in \Omega : \sigma(i) \neq i\}$ is finite. It is easy to see that finitary permutations form a subgroup of $S(\Omega)$ which will be denoted by $S_{fin}(\Omega)$. Finally, let $A_{fin}(\Omega)$ be the subgroup of even permutations in $S_{fin}(\Omega)$ (note that it makes sense to talk about even permutations in $S_{fin}(\Omega)$, but not in $S(\Omega)$).

(a) Prove that the group $A_{fin}(\Omega)$ is simple and that $A_{fin}(\Omega)$ is a subgroup of index two in $S_{fin}(\Omega)$. **Hint:** To prove the first assertion solve problem 5 in [DF, page 151].

(b) Prove that $A_{fin}(\Omega)$ and $S_{fin}(\Omega)$ are both normal in $S(\Omega)$.

(c) Prove that neither of the groups $S(\Omega)$ and $S_{fin}(\Omega)$ is finitely generated. **Hint:** The two groups are not finitely generated for completely different reasons.

(d) Construct a finitely generated subgroup G of $S(\Omega)$ which contains $S_{fin}(\Omega)$. **Note:** This example shows that a subgroup of a finitely generated group does not have to be finitely generated.