Homework #4.

Approximate plan for next week: Direct and semi-direct products and further applications of Sylow theorems (5.4, 5.5).

Problems, to be submitted by Thursday, September 27th

1. Let G be a finite group, P a Sylow p-subgroup of G for some p and H a subgroup of G such that $N_G(P) \subseteq H \subseteq G$. Prove that $N_H(P) = N_G(P)$ and $[G : H] \equiv 1 \mod p$.

Definition: A group G is called *simple* if it does not have a proper non-trivial normal subgroup, that is, the only normal subgroups of G are $\{1\}$ and G.

2. Prove that a group of order $48 = 16 \cdot 3$ is not simple.

3. Prove that a group of order $132 = 3 \cdot 4 \cdot 11$ is not simple.

4. Complete the proof from the last example in Lecture 8: a group G of order 24 has a normal 2-Sylow subgroup or a normal 3-Sylow subgroup or is isomorphic to S_4. Write down the entire argument, including the part we discussed in class (preferably with additional details).

5. Let $\mathbb{F}_p$ be a finite field of order p, let $G = GL_n(\mathbb{F}_p)$ and $P = U_n(\mathbb{F}_p)$, the subgroup of upper-unitriangular matrices in $GL_n(\mathbb{F}_p)$.

(a) Prove that P is a Sylow p-subgroup of G (you may use the formula for $|G|$ established in Homework#2).

(b) Let $B = UT_n(\mathbb{Z}_p)$, the subgroup of upper-triangular matrices in G. Prove that $N_G(P) = B$. **Hint:** To prove the inclusion $N_G(P) \subseteq B$ consider the natural action of G on $X = \mathbb{F}_p^n$. First compute $X^P = \{x \in X : gx = x \text{ for any } g \in P\}$, the set of fixed points of P. Now if $g \in N_G(P)$, what can you say about $g \cdot X^P$ (the image of X^P under the action of g). Use this result and induction on n to prove that $N_G(P) \subseteq B$.

(c) Find $n_p(G)$, the number of Sylow p-subgroups of G.

6. Explicitly describe a Sylow p-subgroup of S_{p^2}. See next page for a hint.

Hint for 6: The largest power of p dividing $(p^2)!$ is p^{p+1}. First find a subgroup of order p^p in S_{p^2} (this is quite easy) and then think how to enlarge it to a subgroup of order p^{p+1}.