Homework #3, due on Thu, Sep 20th, in class

1. An action of a group G on a set X is called *transitive* if it has just one orbit, that is, for any $x, y \in X$ there exists $g \in G$ with $g.x = y$.
(a) Let $(G, X, .)$ be a group action. Prove that if $x, y \in X$ lie in the same orbit, then their stabilizers $Stab_G(x)$ and $Stab_G(y)$ are conjugate, that is, there exists $g \in G$ with $gStab_G(x)g^{-1} = Stab_G(y)$.
(b) Suppose that $(G, X, .)$ is a transitive action and fix $x \in X$. Prove that the kernel of this action is equal to $\bigcap\limits_{g \in G} gStab_G(x)g^{-1}$
(c) Now suppose that G and X are both finite, $(G, X, .)$ is a transitive faithful action (where 'faithful' means the kernel is trivial) and G is abelian. Prove that for any $g \in G \setminus \{1\}$ the fixed set X^g is empty. Deduce that $|X| = |G|$.
Hint: Use (b).

2. Let C be the cube in $\mathbb{R}^3$ whose vertices have coordinates $(\pm 1, \pm 1, \pm 1)$. Let G be the group of rotations of C, that is rotations in $\mathbb{R}^3$ which preserve the cube (you may assume that G is a group without proof). Let X be the set of 4 main diagonals of C (diagonals connecting the opposite vertices). Note that G naturally acts on X and therefore we have a homomorphism $\pi : G \to Sym(X) \cong S_4$. Prove that π is an isomorphism.
Hint: First show that G acts transitively on the 8 vertices of C. Then show that the stabilizer of a fixed vertex had order ≥ 3. This implies that $|G| \geq 24 = |S_4|$. Finally, show that π is injective (since $|G| \geq |S_4|$, this would force π to be an isomorphism).

3. Let G be a group. For each $g \in G$ let $\iota_g : G \to G$ be the conjugation by g, that is, $\iota_g(x) = gxg^{-1}$. Recall that $\iota_g \in \mathrm{Aut}(G)$ for any $g \in G$ and the mapping $\iota : G \to \mathrm{Aut}(G)$ given by $\iota(g) = \iota_g$ is a homomorphism. Elements of the subgroup $\mathrm{Inn}(G) = \iota(G)$ of $\mathrm{Aut}(G)$ are called inner automorphisms.
(a) Prove that for any $g \in G$ and $\sigma \in \mathrm{Aut}(G)$ one has $\sigma \iota_g \sigma^{-1} = \iota_{\sigma(g)}$. Deduce that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
(b) Let H be a normal subgroup of G. Note that for each $g \in G$, the mapping ι_g restricted to H is an automorphism of H. By slight abuse of notation we denote this automorphism of H by ι_g as well. Prove that ι_g is an inner automorphism of H if and only if $g \in H \cdot C_G(H)$ where $C_G(H)$ is the centralizer of H in G.

4. Let $n \geq 4$ and $f = (1,2)(3,4) \in S_n$. Prove that $|C_{S_n}(f)| = 8(n-4)!$. Then describe elements of this centralizer explicitly. **Hint:** What is the conjugacy class of f?

5. (optional) *Necklace-counting problem:* Suppose that we want to build a necklace using n beads of k possible colors (we do not have to use all available colors). Two necklaces will be considered equivalent if they can be obtained from each other using rotations or reflections. What is the number of non-equivalent necklaces one can construct?

Approach using group actions. Let X be the set of all necklaces with beads of k possible colors located at the vertices of a regular n-gon. The dihedral group D_{2n} has a natural action on X, and the orbits under that action are precisely equivalence classes of necklaces in the above sense. Use this interpretation and Burnside's orbit-counting formula to prove that for $n = 9$ the number of non-equivalent necklaces is

$$\frac{k^9 + 2k^3 + 6k + 9k^5}{18}.$$

Then try to do the same for general n.

6. Problem 10 on page 117 in Dummit and Foote.

7. Let G be a group such that $G/Z(G)$ is cyclic. Prove that G is abelian.