Homework #11.

Plan for next week: Proof of the Nullstellensatz (Dec 4). Exact sequences (Dec 6).

Problems, to be submitted by Thursday, Dec 6

1. Let p be a prime. Use direct counting argument to find the number of monic irreducible polynomials of degree n in $\mathbb{F}_p[x]$ for $n = 2, 3, 4$ and check that your answer matches the general formula derived in the online supplement

`http://people.virginia.edu/~mve2x/7751_Fall2011/irreducible.pdf`

Hint: The number of irreducible monic polynomials of degree n equals the total number of monic polynomials of degree n minus the number of reducible monic polynomials of degree n; the latter can be computed considering possible factorizations into irreducibles (assuming the number of irreducible monic polynomials of degree m for $m < n$ has already been computed).

2. Give an example of a domain R (other than a field or the zero ring) which has no irreducible elements. **Hint:** Start with the ring of power series $R = F[[x]]$ where F is a field. Then up to associates x is the only irreducible element of R. Construct a larger ring $R_1 \supseteq R$ s.t. x is reducible in R_1, but $R_1 \cong F[[x]]$. Then iterating the process construct an infinite ascending chain $R \subseteq R_1 \subseteq R_2 \subseteq \ldots$ and consider its union.

3. Let R be a commutative Noetherian ring and $\varphi : R \to R$ a surjective ring homomorphism. Prove that φ must be an isomorphism. **Hint:** Consider the ideals $\operatorname{Ker}(\varphi^n)$, $n \in \mathbb{N}$, where φ^n is φ composed with itself n times.

4. Let R be a commutative Noetherian ring. Prove that the ring $R[[x]]$ of power series over R is also Noetherian. **Hint:** As you may expect, this can be proved similarly to the Hilbert basis theorem (HBT) except that you have to consider the lowest degree terms, not the highest degree terms (which may not exist). In fact, the first part of the proof is even easier than in HBT, but you will need some kind of limit argument at the end.

5. Let k be a field, n a positive integer and k^n the n-dimensional affine space over k. Recall that a non-empty algebraic subset $V \subseteq k^n$ is called *irreducible* if V cannot be represented as a union $V = V_1 \cup V_2$ where $V_1 \neq V$, $V_2 \neq V$

and V_1 and V_2 are both algebraic. Prove that V is irreducible if and only if its vanishing ideal $I(V)$ is prime.
Hint: First establish the following properties (they follow very easily from what we discussed in class):

(i) If W_1 and W_2 are algebraic subsets with W_1 strictly contained in W_2, then $I(W_1)$ strictly contains $I(W_2)$.

(ii) If W is an algebraic subset, $I = I(W)$ and J is an ideal which strictly contains I, then $Z(J)$ is strictly contained in W.