Homework #10.

Plan for next week: Finite fields (9.5 + some other stuff), Hilbert basis theorem (9.6).

Problems, to be submitted by Thursday, November 29th

1. Let D be a positive integer such that $D \equiv 3 \mod 4$, and let $R = \mathbb{Z}[\frac{1+\sqrt{-D}}{2}]$, that is, R is the minimal subring of $\mathbb{C}$ containing $\mathbb{Z}$ and $\frac{1+\sqrt{-D}}{2}$.

(a) Prove that $R = \{a + b\frac{1+\sqrt{-D}}{2} : a, b \in \mathbb{Z}\}$. You may skip details, but it should be clear from your argument where the assumption $D \equiv 3 \mod 4$ is used (otherwise the result is simply not true).

(b) Assume that $D = 3, 7$ or 11. Prove that R is a Euclidean domain.

2. Let $R = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$. Find an element of R which is irreducible but not prime and deduce that R is not a unique factorization domain (UFD).

Hint: Consider the equality $2 \cdot 2 = (\sqrt{5}+1)(\sqrt{5}-1)$. In order to check whether some element of R is irreducible it is convenient to use the standard norm function $N : R \to \mathbb{Z}_{\geq 0}$ given by $N(a + b\sqrt{5}) = |a^2 - 5b^2|$ (note that $N(uv) = N(u)N(v)$).

3. Let $R = \mathbb{Z} + x\mathbb{Q}[x]$, the subring of $\mathbb{Q}[x]$ consisting of polynomials whose constant term is an integer.

(a) Show that the element αx, with $\alpha \in \mathbb{Q}$ is NOT irreducible in R. Then show that x cannot be written as a product of irreducibles in R. Note that by Lecture 21, this implies that R is not Noetherian.

(b) Now prove directly that R is not Noetherian by showing that $I = x\mathbb{Q}[x]$ is an ideal of R which is not finitely generated.

(c) Give an example of a non-Noetherian domain which is a UFD.

4. Let F be a field, take $f(x, y) \in F[x, y]$, and write $f(x, y) = \sum_{i=0}^{n} c_i(y)x^i$ where $c_i(y) \in F[y]$. Suppose that

(i) There exists $\alpha \in F$ such that $c_n(\alpha) \neq 0$

(ii) $gcd(c_0(y), c_1(y), \ldots, c_n(y)) = 1$ in $F[y]$

(iii) $f(x, \alpha)$ is an irreducible element of $F[x]$ (where $f(x, \alpha)$ is the polynomial obtained from $f(x, y)$ be substituting α for y).

Prove that $f(x, y)$ is irreducible in $F[x, y]$.

5. Prove that the following polynomials are irreducible:

(a) $f(x, y) = y^3 + x^2y^2 + x^3y + x^2 + x$ in $\mathbb{Q}[x, y]$

(b) $f(x, y) = xy^2 + x^2y + 2xy + x + y + 1$ in $\mathbb{Q}[x, y]$

(c) $f(x) = x^5 - 3x^2 + 15x - 7$ in $\mathbb{Q}[x]$

Hint: For (a) and (b) – think of $\mathbb{Q}[x, y]$ as $(\mathbb{Q}[x])[y]$, the ring of polynomials in one variable y over $R = \mathbb{Q}[x]$, or as $(\mathbb{Q}[y])[x]$. For (c): By Gauss Lemma, it is enough to prove irreducibility of $f(x)$ in $\mathbb{Z}[x]$. Consider the reduction map $u(x) \to \overline{u}(x)$ from $\mathbb{Z}[x]$ to $\mathbb{Z}_3[x]$, consider possible factorizations of $\overline{f}(x)$ and show that none of them can be lifted to a factorization of $f(x)$ (the general idea is similar to the proof of the Eisenstein criterion).