Homework #8.

Plan for next week: Properties of ideals (7.4). Ring of fractions and Localization (7.5).

Problems, to be submitted by Thursday, October, 27th

1. (a) Let X be a finite set with $|X| = n$, and let $F = F(X)$ be the (standard) free group on X. Prove that $F/[F,F] \cong \mathbb{Z}^n$. **Hint:** First show that there is a natural epimorphism $\pi : F/[F,F] \to \mathbb{Z}^n$ and then argue that π must be an isomorphism.
(b) Let X and Y be finite sets. Prove that free groups $F(X)$ and $F(Y)$ are isomorphic if and only if $|X| = |Y|$.

2. Let X be a set and $F(X)$ the free group on X. Given $f \in F(X)$, the *length of* f, denoted by $l(f)$ is defined to be the length of the unique reduced word in $X \cup X^{-1}$ representing f. Equivalently, $l(f)$ is the smallest n such that $f = x_1^{\epsilon_1} \ldots x_n^{\epsilon_n}$ for some $x_i \in X$ and $\epsilon_i \in \{\pm 1\}$.
(a) Prove that for any $f \in F(X)$ there exists integers $a, b \in \mathbb{Z}_{\geq 0}$ (depending on f) such that $l(f^n) = na + 2b$ for any $n \in \mathbb{N}$. Describe explicitly (i.e. give an algorithm) how to compute a and b for a given f.
(b) Use (a) to show that free groups are torsion-free.

3. Let $G = \langle x, y \mid x^2, y^2 \rangle$. Prove that the element $xy \in G$ has infinite order. **Hint:** Use von Dyck's theorem.

4. (a) Explain why for any $n \in \mathbb{N}$ there are only finitely many isomorphism classes of groups of order n.
(b) Let G be a finitely generated group and H a finite group. Prove that there are only finitely many homomorphisms from G to H. **Hint:** A homomorphism from G is completely determined by its values on generators.
(c) Let G be a finitely generated group. Prove that for any $n \in \mathbb{N}$ there are only finitely many normal subgroups of index n in G. Then deduce that G has only finitely many subgroups of index n (use small index lemma).

5. A group G is called *residually finite* if for any distinct elements $x, y \in G$ there exists a finite group H and a homomorphism $\phi : G \to H$ such that $\phi(x) \neq \phi(y)$. Thus, informally speaking, a group is residually finite if its elements can be separated via their images in finite quotients of G.

Clearly, any finite group G is residually finite (in which case for any $x, y \in G$ we take $H = G$ and ϕ the identity mapping). Probably, the simplest example of an infinite residually finite group is $\mathbb{Z}$: if $x, y \in \mathbb{Z}$ are distinct, choose any $n \in \mathbb{N}$ such that $n \nmid (x - y)$, and let $\phi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the natural projection; then clearly $\phi(x) \neq \phi(y)$.

(a) Prove that the following conditions on a group G are equivalent:

(i) G is residually finite

(ii) For any non-identity element $x \in G$ there exists a finite group H and a homomorphism $\phi : G \to H$ such that $\phi(x) \neq 1$.

(iii) The intersection of all normal subgroups of finite index in G is trivial.

(b) Prove that a subgroup of a residually finite group is residually finite.
(c) Let $n \geq 2$ be an integer. Prove that the group $GL_n(\mathbb{Z})$ is residually finite.
Hint: Probably, it is the easiest to show that condition (ii) from part (a) holds.
Note: It is well known that free groups of finite rank can be embedded in $SL_2(\mathbb{Z})$. Thus, (b) and (c) imply that free groups of finite rank are residually finite (but there are other interesting proofs of that fact).

6. (optional) Recall from Homework#3 that a group G is called hopfian if any surjective homomorphism $\phi : G \to G$ is injective. Prove that any finitely generated residually finite group is hopfian. **Hint:** Suppose, on the contrary, that there exists a finitely generated group G and a surjective homomorphism $\phi : G \to G$ with $\operatorname{Ker}\phi$ non-trivial. Then $G/\operatorname{Ker}\phi \cong G$. Deduce that for any $n \in \mathbb{N}$ there is a bijection between the set of all normal subgroups of index n in G and the set of those normal subgroups of index n in G which contain $\operatorname{Ker}\phi$. Then use Problem 4 to reach a contradiction with the assumption that G is residually finite.