Homework #7.

Plan for next week: Free groups (continued) and presentations of groups by generators and relators (§ 6.3), start ring theory (§ 7.1).

Problems, to be submitted by Thursday, October, 20th

1. Let $n \geq 3$ and $G = D_{2n}$, the dihedral group of order $2n$.
(a) Prove that G contains a subgroup isomorphic to D_{2k} for any $k \mid n$.
(b) Prove that the dihedral group G is nilpotent if and only if n is power of 2.

2. (a) Let R be an associative ring with 1, and let $a, b \in R$ be such that $1+a$ and $1+b$ are invertible. Prove the following formula

$$(1+a)^{-1}(1+b)^{-1}(1+a)(1+b) = 1 + (1+a)^{-1}(1+b)^{-1}(ab - ba).$$

(b) Let R be an associative ring with 1 and $n \geq 2$ be an integer, and let $U_n(R)$ be the upper unitriangular subgroup of $GL_n(R)$. Prove that $U_n(R)$ is niplotent of class $n-1$ (we briefly outlined the proof in class). Note that you will need to apply (a) not to R itself but to the ring of $n \times n$ matrices over R.

3. (a) Let $n \in \mathbb{N}$ be an integer, and suppose that for every non-prime divisor m of n there are no simple groups of order m. Prove that any group of order n is solvable.

(b) Prove that any group of order $p^k q$, where $p > q$ are distinct primes, is solvable.

4. Problems 31 and 32 on page 200 of DF. **Note:** Problem 31 follows very easily from Lemma 11.2 (about the structure of minimal normal subgroups).

5. (a) Let A and B be finitely generated groups. Prove that the wreath product $AwrB$ is also finitely generated. **Hint:** Recall that $AwrB = C \rtimes B$ where $C = \oplus_{b\in B} A_b$ (with each $A_b \cong A$). Let S be a generating set for A, T a generating set for B, fix $b \in B$, and let S_b be the image of S under an isomorphism $A \to A_b$. Prove that $S_b \cup T$ generates $AwrB$.

(b) Use (a) to give a simple example showing that a subgroup of a finitely generated group may not be finitely generated.

6. (a) (practice) In class we outlined the proof of the fact that $\mathbb{Z}_p wr \mathbb{Z}_p$ is isomorphic to the Sylow p-subgroup of S_{p^2}. Fill in the details of that proof.

(b) (bonus) Realize the Sylow p-subgroup of S_{p^3} in terms of wreath products and prove your answer (you may skip some technical details).