**Homework #6.**

**Plan for next week:** Free groups (6.3)

**Problems, to be submitted by Thursday, October, 13th**

**1.** (a) Classify all abelian groups of order $360 = 2^3 \cdot 3^2 \cdot 5$ up to isomorphism. For each isomorphism type, state the corresponding elementary divisors form and invariant factors form.
(b) Let $n \in \mathbb{N}$, and decompose $n$ as a product of primes: $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$. Find the number of non-isomorphic abelian groups of order $n$. Express your answer in terms of the partition function.

**2.** Let $G$ be a finite abelian group. Prove that $G$ is cyclic if and only if $G$ does not contain a subgroup isomorphic to $B \oplus B$ for some non-trivial group $B$.

**3.** Let $G$ be an abelian group (not necessarily finitely generated), and let $Tor(G)$ be the set of elements of finite order in $G$. Recall that $Tor(G)$ is a subgroup of $G$ (since $G$ is abelian), called the torsion subgroup of $G$.
(a) Prove that the quotient group $G/Tor(G)$ is torsion-free (a group is called torsion-free if it does not have non-identity elements of finite order).
(b) For each prime $p$ let $Tor_p(G)$ be the set of elements of order $p^k$ (with $k \geq 0$) in $G$. Prove that each $Tor_p(G)$ is a subgroup of $Tor(G)$ and that $Tor(G)$ is a direct sum of these subgroups where $p$ runs over all primes.

**4.** Let $\Omega$ be a countable set (for simplicity you may assume that $\Omega = \mathbb{Z}$, the integers). Let $S(\Omega)$ be the group of all permutations of $\Omega$. A permutation $\sigma \in S(\Omega)$ is called *finitary* if it moves only a finite number of points, that is, the set $\{i \in \Omega : \sigma(i) \neq i\}$ is finite. It is easy to see that finitary permutations form a subgroup of $S(\Omega)$ which will be denoted by $S_{fin}(\Omega)$. Finally, let $A_{fin}(\Omega)$ be the subgroup of even permutations in $S_{fin}(\Omega)$ (note that it makes sense to talk about even permutations in $S_{fin}(\Omega)$, but not in $S(\Omega)$).
(a) Prove that the group $A_{fin}(\Omega)$ is simple and that $A_{fin}(\Omega)$ is a subgroup of index two in $S_{fin}(\Omega)$. **Hint:** To prove the first assertion solve problem 5 in [DF, page 151].
(b) Prove that $A_{fin}(\Omega)$ and $S_{fin}(\Omega)$ are both normal in $S(\Omega)$.
(c) Prove that neither of the groups $S(\Omega)$ and $S_{fin}(\Omega)$ is finitely generated. **Hint:** The two groups are not finitely generated for completely different

reasons.
(d) Construct a finitely generated subgroup $G$ of $S(\Omega)$ which contains $S_{fin}(\Omega)$.
**Note:** This example shows that a subgroup of a finitely generated group does not have to be finitely generated.