29. Affine algebraic sets, geometric interpretation of HBT and Nullstellensatz

29.1. **Affine algebraic sets.** Let k be a field. For $n \in \mathbb{N}$ let k^n be the n-dimensional affine space over k. We shall "think" of elements of k^n as points, not vectors. Any $f \in k[x_1, \ldots, x_n]$ naturally defines a function on k^n.

Definition. Let S be a subset of $k[x_1, \ldots, x_n]$. The set

$$Z(S) = \{(a_1, \ldots, a_n) \in k^n : f(a_1, \ldots, a_n) = 0 \text{ for any } f \in S\}$$

is called the zero set of S.

Remark: The zero set of $S \subset k[x_1, \ldots, x_n]$ is often called the *vanishing set* of S.

Example:

(i) $Z(\{1\}) = \emptyset$

(ii) $Z(\{0\}) = k^n$

(iii) Let $n = 4$ and $S = \{x_2^2 - x_1^2, x_3 - x_1^2\}$. Then $Z(S) = \{(a_1, \pm a_1, a_1^2, a_4)\}$ is a union of two parabolic cylinders.

Definition. A subset $V \subseteq k^n$ is called algebraic if $V = Z(S)$ for some (possibly infinite) set of polynomials S.

If Y is any subset of k^n, we can consider

$$I(Y) = \{f \in k[x_1, \ldots, x_n] : f(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in V\}.$$

Clearly, $I(Y)$ is an ideal of Y called the vanishing ideal of Y.

Lemma 29.0. *The maps $Y \mapsto I(Y)$ and $S \mapsto Z(S)$ are inclusion-reversing, that is,*

(a) *If $Y_1 \subseteq Y_2 \subseteq k^n$, then $I(Y_1) \supseteq I(Y_2)$;*

(b) *If $S_1 \subseteq S_2 \subseteq k[x_1, \ldots, x_n]$, then $Z(S_1) \supseteq Z(S_2)$.*

Proof. Obvious. □

Lemma 29.1. *Let S be a subset of $k[x_1, \ldots, x_n]$ and Y a subset of k^n. Then*

(a) *$I(Z(S)) \supseteq S$;*

(b) *$Z(I(Y)) \supseteq Y$;*

(c) *$Z(S) = Z(J)$ where $J = (S)$ is the ideal generated by S;*

(d) *Y is algebraic $\iff$ $Z(I(Y)) = Y$.*

Proof. (a), (b), (c) and the "$\Leftarrow$" part of (d) are clear. Let us prove the "$\Rightarrow$" part of (d). Suppose that Y is algebraic, so $Y = Z(S)$ for some S. Then $I(Y) = I(Z(S) \supseteq S$ by (a). By Lemma 29.0(b) we have $Z(I(Y)) \subseteq Z(S) = Y$. The opposite inclusion $Z(I(Y)) \supseteq Y$ holds by (b). $\square$

29.2. **Geometric interpretation of HBT.** Let $V \subseteq k[x_1, \ldots, x_n]$ be an algebraic subset. By HBT $I(V) = (f_1, \ldots, f_n)$ for some $f_1, \ldots, f_n$. By Lemma 29.1(c)(d) we have

$$V = Z(I(V)) = Z((f_1, \ldots, f_n)) = Z(\{f_1, \ldots, f_n\})$$

Thus, V is the zero set of a finite family of polynomials:

Corollary 29.2. *Every algebraic set in k^n is the set of common zeroes of a finite family of polynomials.*

Corollary 29.3. *Every descending chain of algebraic subsets of k^n must stabilize after finitely many steps.*

Proof. Suppose we have a chain $V_1 \supset V_2 \supset \ldots$ where each V_i is algebraic and all inclusions are strict. Then $I(V_1) \subseteq I(V_2) \subseteq \ldots$, so $I(V_m) = I(V_{m+1})$ for some m since the ring $k[x_1, \ldots, x_n]$ is Noetherian.
But then $Z(I(V_m)) = Z(I(V_{m+1}))$, and thus $V_m = V_{m+1}$ by Lemma 29.1(d) contrary to our assumption. $\square$

29.3. **Radical ideals and Nullstellensatz.** Notice that we established an inclusion-reversing correspondence

$$\{\text{algebraic subsets of } k^n\} \longleftrightarrow \text{ ideals of } k[x_1, \ldots, x_n] \tag{29.1}$$

$$\begin{array}{ccc} V & \mapsto & I(V) \\ Z(J) & \leftarrow & J \end{array}$$

We already saw that $Z(I(V)) = V$ if V is algebraic. Is it true that $I(Z(J)) = J$ for any ideal J? Positive answer would have implied that the above correspondence is bijective, but the answer turns out to be 'No':
Example: 1. Let k be any field, $n = 1$ and $x = x_1$. Choose any $a \in k$ and let $J = ((x - a)^2)$. Then $Z(J) = \{a\}$, but $I(Z(J)) = I(\{a\}) = (x - a)$ is strictly larger than J.
2. Once again let $n = 1$, $x = x_1$, and suppose that k is not algebraically closed. Then there exists a non-constant polynomial $f(x) \in k[x]$ without any roots. Let $J = (f(x))$. Then $Z(J) = \emptyset$, and so $I(Z(J)) = k \neq J$.

It turns out that if k is algebraically closed, then Example 1 illustrates "the only reason" why the equality $Z(I(J)) = J$ may fail to be true. In this case we can make the correspondence in (29.1) bijective by restricting from all ideals to radical ideals defined below.

Definition. Let R be a commutative ring and J an ideal of R. The set $\sqrt{J} := \{r \in R : r^n \in J \text{ for some } n \in \mathbb{N}\}$ is called the radical of J.

It is easy to show that $\sqrt{J}$ is also an ideal. Also note that $J \subseteq \sqrt{J}$ and $\sqrt{\sqrt{J}} = \sqrt{J}$.

Definition. An ideal J of a ring R is called radical if $\sqrt{J} = J$.

It is easy to see that J is radical $\iff$ R/J has no nilpotent elements.

Theorem (Hilbert's Nullstellensatz)**.** *If k is an algebraically closed field and J is an ideal of $k[x_1, \ldots, x_n]$, then $I(Z(J)) = \sqrt{J}$. In particular,*

$$I(Z(J)) = \sqrt{J} \iff J \text{ is radical.}$$

Hopefully we will have time to prove Nullstellensatz next semester.

Corollary 29.4. *If k is an algebraically closed field,* (29.1) *yields a bijective correspondence between algebraic subsets of k^n and radical ideals of $k[x_1, \ldots, x_n]$.*

Note that every prime ideal is radical. It is not hard to show (New Year Homework, Problem#3) that under the above bijection prime ideals correspond to irreducible algebraic subsets.

Definition. An algebraic subset $V \subseteq k^n$ is called irreducible if $V \neq \emptyset$ and V cannot be written as the union $V = V_1 \cup V_2$ where V_1 and V_2 are both algebraic, with $V_1 \neq V$ and $V_2 \neq V$.

Finally, Hilbert's Nullstellensatz easily implies that maximal ideals of $k[x_1, \ldots, x_n]$ are in bijective correspondence with points in k^n (once again, assuming that k is algebraically closed) – a point $(a_1, \ldots, a_n)$ corresponds to the ideal $(x_1 - a_1, \ldots, x_n - a_n)$. It is easy to see that each of those ideals is maximal; the non-trivial part which requires Nullstellensatz is that there are no other maximal ideals.