26. Completions of rings

26.1. **Norms on rings.** Let A be a ring and suppose that we have a function $\| \ \| : A \to \mathbb{R}_{\geq 0}$, called a <u>norm</u>, such that

(1) $\|x\| = 0 \iff x = 0$
(2) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in A$
(3) $\|xy\| \leq \|x\| \cdot \|y\|$ for all $x, y \in A$

Define the metric d on A associated with the norm $\| \ \|$ by setting $d(x, y) = \|x - y\|$. This is indeed a metric by conditions (1) and (2). With this metric A becomes a topological ring, that is, A is a topological space and the mappings $(x, y) \mapsto x + y$ and $(x, y) \mapsto x \cdot y$ from $A \times A$ to A are continuous. Let $\widehat{A}$ be the ring-theoretic completion of A with respect to d. As a set $\widehat{A}$ is just the usual topological completion of A, that is,

$$\widehat{A} = \{\text{equivalence classes of Cauchy sequences of elements of} A\}$$

Ring operations on $\widehat{A}$ are defined by setting

$$[x_n] + [y_n] = [x_n + y_n] \quad \text{and} \quad [x_n] \cdot [y_n] = [x_n \cdot y_n]$$

where $[x_n]$ is the equivalence class of the sequence $\{x_n\}$.
Conditions (2) and (3) in the definition of a norm ensure that addition and multiplication on $\widehat{A}$ are well defined and continuous.

Remark: (a) There is a natural embedding $\iota : A \to \widehat{A}$ where $\iota(x)$ is the class of the constant sequence $x, x, \ldots$. From now on we shall identify A with $\iota(A)$.
(b) The norm $\| \ \|$ can be extended from A to $\widehat{A}$ by setting

$$\| \, [x_n] \, \| = \lim_{n \to \infty} \|x_n\|.$$

It is easy to see that this extension still satisfies the axioms (1)-(3).
Example: Let $A = \mathbb{Q}$ (rationals) and $\|x\| = |x|$ (the absolute value of x). Then $\widehat{A} = \mathbb{R}$ (reals).

26.2. **Completion with respect to I-adic metric.** Let A be a ring with 1 and fix an ideal I of A. Recall that for each $n \in \mathbb{N}$ we have

$$I^n = \{\text{finite sums of elements of the form } i_1 \ldots i_n : i_k \in I\}.$$

We also set $I^0 = A$. Assume that

$$\bigcap_{n \in \mathbb{N}} I^n = \{0\}.$$

Then given $0 \neq x \in A$, there exists unique $n \in \mathbb{Z}_{\geq 0}$ such that $x \in I^n \setminus I^{n+1}$. This n will be denoted by $\deg(x)$ and called the degree of x. We also set $\deg(0) = \infty$. Define

$$\|x\| = 2^{-\deg(x)}$$

We claim that $\| \quad \|$ is a norm on A. Indeed, since each I^n is closed under addition and $I^{n+m} = I^n \cdot I^m$ we have

$$\text{(a)} \ \deg(x+y) \geq \min\{\deg(x), \deg(y)\} \quad \text{(b)} \ \deg(xy) \geq \deg(x) + \deg(y)$$

These translate into "opposite" inequalities for $\| \quad \|$

$$\text{(a')} \ \|x+y\| \leq \max\{\|x\|, \|y\|\} \qquad \text{(b')} \ \|xy\| \leq \|x\| \cdot \|y\|$$

Inequality (a') is called the ultrametric triangle inequality and is stronger than the usual triangle inequality $\|x+y\| \leq \|x\| + \|y\|$.
Conditions (a') and (b') and the fact that each nonzero element of A has finite degree imply that $\| \quad \|$ is indeed a norm. It is called the I-adic norm on A. The associated metric and topology are called the I-adic metric (resp. topology).

Definition. The completion of A with respect to the I-adic norm is called the I-adic completion of A and will be denoted by $\widehat{A}_I$.

Example 1: Let R be a commutative ring, $A = R[x]$ and $I = (x)$. We claim that

$$\widehat{A}_I \cong R[[x]] \quad \text{(power series over } R\text{)}.$$

Proof (sketch). Consider two arbitrary elements of $R[x]$

$$f(x) = r_0 + \ldots + r_n x^n \quad \text{ and } \quad g(x) = s_0 + \ldots + s_n x^n.$$

Then

$$\|f - g\| = 2^{-m} \text{ where } m \text{ is the } \textbf{minimal} \text{ integer for which } r_m \neq s_m \quad (*)$$

Define the map

$$\Phi : \{\text{Cauchy sequences in } R[x]\} \to R[[x]]$$

as follows:
Let $f_1, f_2, \ldots$ be a Cauchy sequence in $R[x]$. Let f_{ik} be the coefficient of x^k in f_i. By (*) for any $k \in \mathbb{Z}_{\geq 0}$ the sequence $\{f_{ik}\}_{i=1}^{\infty}$ is eventually constant, that is, there exists $N = N(k) \in \mathbb{N}$ and $r_k \in R$ such that $f_{ik} = r_k$ for $i > N$. Define

$$\Phi([f_1, f_2, \ldots]) = r_0 + r_1 x + \ldots$$

Straightforward verification shows that Φ is well defined and establishes an isomorphism between $\widehat{A}_I$ and $R[[x]]$. $\square$

Example 2: Let $A = \mathbb{Z}$, l an integer ≥ 2 and $I = (l)$. Then $\widehat{A}_I$ is called the ring of <u>l-adic integers</u> and denoted by $\widehat{\mathbb{Z}}_l$.

Claim 26.1. *The following hold:*

(1) *For any sequence of integers $a_0, a_1, \dots$ the series $a_0 + a_1 l + a_2 l^2 + \dots$ converges in $\widehat{\mathbb{Z}}_l$.*
(2) *Any $x \in \widehat{\mathbb{Z}}_l$ can be uniquely written as $x = \sum\limits_{n=0}^{\infty} a_n l^n$ where each $a_n \in \mathbb{Z}$ and $0 \leq a_n \leq l - 1$.*

Proof (sketch). (1) is easy. For instance, it follows from the fact that in any metric space satisfying ultrametric triangle inequality a series $\sum\limits_{n=1}^{\infty} c_n$ converges if and only if $\lim\limits_{n \to \infty} c_n = 0$.
(2) can be proved similarly to example 1. A key fact one needs to use is that any non-negative integer m can be uniquely written as a (finite) sum $m = \sum\limits_{n=0}^{k} a_n l^n$ with $0 \leq a_n \leq l - 1$ (this is just the expansion of m to the base l). □

An "informal consequence" of Claim 26.1(2) is that as a set $\widehat{\mathbb{Z}}_l$ can be identified with "power series in l." Addition and multiplication on $\widehat{\mathbb{Z}}_l$ can be defined using the usual "carry-over" algorithm for adding and multiplying integers written to the base l.