New Year Homework.

1. Let R be a commutative Noetherian ring and $\varphi : R \to R$ a surjective ring homomomorphism. Prove that φ must be an isomorphism. **Hint:** Consider the ideals $\operatorname{Ker}(\varphi^n)$, $n \in \mathbb{N}$, where φ^n is φ composed with itself n times.

2. Let R be a commutative Noetherian ring. Prove that the ring $R[[x]]$ of power series over R is also Noetherian. **Hint:** As you may expect, this can be proved similarly to the Hilbert basis theorem (HBT) except that you have to consider the lowest degree terms, not the highest degree terms (which may not exist). In fact, the first part of the proof is even easier than in HBT, but you will need some kind of limit argument at the end.

3. Let k be a field, n a positive integer and k^n the n-dimensional affine space over k. Recall that a non-empty algebraic subset $V \subseteq k^n$ is called *irreducible* if V cannot be represented as a union $V = V_1 \cup V_2$ where $V_1 \neq V$, $V_2 \neq V$ and V_1 and V_2 are both algebraic. Prove that V is irreducible if and only if its vanishing ideal $I(V)$ is prime.
Hint: First establish the following properties (they follow very easily from what we discussed in class):

(i) If W_1 and W_2 are algebraic subsets with W_1 strictly contained in W_2, then $I(W_1)$ strictly contains $I(W_2)$.

(ii) If W is an algebraic subset, $I = I(W)$ and J is an ideal which strictly contains I, then $Z(J)$ is strictly contained in V.

4. Let A be a commutative ring with 1, let I be an ideal of A, and assume that $\cap_{n\in\mathbb{N}} I^n = \{0\}$. Let $\widehat{A}_I$ be the I-adic completion of A. Prove that if I is a maximal ideal, then $\widehat{A}_I$ is a local ring whose unique maximal ideal is the closure of I in $\widehat{A}_I$.
Hint: Let $\widehat{I}$ be the closure of I in $\widehat{A}_I$. Show that

(a) If x is an element of $A \setminus I$, then x is invertible in $\widehat{A}_I$.

(b) $\widehat{I}$ is an ideal of $\widehat{A}_I$.

(c) If $\{x_n\}$ is a convergent sequence of elements of $A \setminus I$, then the sequence $\{x_n^{-1}\}$ of their inverses in $\widehat{A}_I$ also converges and $\lim x_n^{-1} = (\lim x_n)^{-1}$.

(d) Deduce from (a) and (c) that any element of $\widehat{A}_I \setminus \widehat{I}$ is invertible in $\widehat{A}_I$.

5. Let p be a prime, $\mathbb{F}_p$ a finite field of order p and $\mathbb{Z}_p$ the ring of p-adic integers.

(a) Prove the following version of Hensel's lemma:

Hensel's lemma: Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients and $\overline{f}(x) \in \mathbb{F}_p[x]$ be the reduction of $f(x) \mod p$. Suppose that there exists $a \in \mathbb{Z}$ such that $\overline{f}(a) = 0$ and $\overline{f}\,'(a) \neq 0$. Prove that f has a root in $\mathbb{Z}_p$.

Hint: Prove by induction that there exists a sequence of integers $a = a_1, a_2, \ldots$ such that $p^n | f(a_n)$, $p \nmid f'(a_n)$ and $a_{n+1} = a_n + p^n b_n$ for some $b_n \in \mathbb{Z}$.

(b) Assume that p is odd. Prove that the equation $x^2 = -1$ has a solution in $\mathbb{Z}_p$ if and only if $p \equiv 1 \mod 4$. What happens when $p = 2$?