Homework #9.

Plan for next week: Unique factorization domains (8.3 and 9.3 in DF).

Problems, to be submitted by Thursday, November 12th

1. (a) Let G be a finitely generated group. Use Zorn's lemma to show that G has a maximal subgroup (recall that a maximal subgroup of G is a maximal element of the set of proper subgroups of G partially ordered by inclusion). **Hint:** The key step is to show that if $\mathcal{C}$ is a chain of proper subgroups of G, then the union of subgroups in this chain is not the whole G.
(b) (optional) If your proof in (a) is correct, a nearly identical argument should imply that G always has a maximal normal subgroup. However, the latter is true under weaker assumptions on G. Can you find a natural condition on G (weaker than finite generation) that guarantees the existence of a maximal normal subgroup? Can you give an example of a group which has a maximal normal subgroup, but no maximal subgroup?

2. Let R be a commutative ring with 1. The *nilradical* of R denoted $Nil(R)$ is the set of all nilpotent elements of R, that is

$$Nil(R) = \{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}.$$

The *Jacobson radical* of R denoted by $J(R)$ is the intersection of all maximal ideals of R. Prove that
(a) $Nil(R)$ and $J(R)$ are ideals of R
(b) $Nil(R) \subseteq J(R)$.

3. Let D be a positive integer such that $D \equiv 3 \mod 4$, and let $R = \mathbb{Z}[\frac{1+\sqrt{-D}}{2}]$, that is, R is the minimal subring of $\mathbb{C}$ containing $\mathbb{Z}$ and $\frac{1+\sqrt{-D}}{2}$.
(a) Prove that $R = \{a + b\frac{1+\sqrt{-D}}{2} : a, b \in \mathbb{Z}\}$. You may skip details, but it should be clear from your argument where the assumption $D \equiv 3 \mod 4$ is used (otherwise the result is simply not true).
(b) Assume that $D = 3, 7$ or 11. Prove that R is a Euclidean domain.

4. Let $R = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$. Find an element of R which is irreducible but not prime and deduce that R is not a unique factorization domain (UFD). **Note:** So far we only proved that irreducible elements are prime in PID. Next week we will show that every PID is a UFD, and that the property *prime=irreducible* holds in all UFDs as well. **Hint:** Consider the equality $2 \cdot 2 = (\sqrt{5}+1)(\sqrt{5}-1)$. In order to check whether some element of R is irreducible it is convenient to use the standard norm function $N : R \to \mathbb{Z}_{\geq 0}$ given by $N(a + b\sqrt{5}) = |a^2 - 5b^2|$ (note that $N(uv) = N(u)N(v)$).

5. (a) (practice) Problem 7.3.34. Note: in all exercises in 7.3 R is assumed to be a ring with 1 (this is crucial for this problem). Also note that IJ is NOT defined to be the set $\{ij : i \in I, j \in J\}$; by definition, IJ is the set of finite sums of elements of the form ij, with $i \in I, j \in J$.
(b) Read the section on the Chinese remainder theorem (7.6).

6. (practice) Let $R = C[0, 1]$ be the ring of continuous functions $f : [0, 1] \to \mathbb{R}$ with pointwise addition and multiplication.
(a) Find a nonzero element $f \in R$ which is neither invertible nor a zero divisor (and prove that your element has these properties).
(b) Fix $a \in [0, 1]$, and let $I = \{f \in R : f(a) = 0\}$. Prove that I is an ideal of R and that I is not principal.