Homework #8.

Plan for next week: Finish free groups and presentations by generators and relations. Start ring theory (perhaps on Tuesday), 7.1-7.4. Sections 7.1-7.3 contain basic material on ring theory covered in undergraduate courses and will only be briefly reviewed – read 7.1-7.3 in DF (or at least browse through these sections) before the class on Thursday Oct 29.

Problems, to be submitted by Thursday, October, 29th

1. Let X be a set. A group F is called a *free abelian group on* X if
(i) F is abelian
(ii) F solves the following universal problem: There exists a mapping $\iota : X \to F$ such that for any ABELIAN group A and any mapping $f : X \to A$ there exists unique homomorphism $f_* : F \to A$ making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\iota} & F \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle f_*} \\ & & A \end{array} \qquad (1)$$

(a) Prove that a free abelian group on X is (at most) unique up to isomorphism, that is, any two groups satisfying (i) and (ii) are isomorphic to each other. **Note:** This is similar to the proof of uniqueness of free groups.
(b) Suppose that X is finite, and let $n = |X|$. Prove that free abelian group on X exists and is isomorphic to $\mathbb{Z}^n$. **Note:** Since $\mathbb{Z}^n$ is abelian, by (a) it is enough to show that $F = \mathbb{Z}^n$ solves the universal problem in (ii). It should be clear from your proof how you use the fact that A in (1) is abelian.
(c) (optional) Show that free abelian group on X exists for any set X and describe it. A common notation for such group is $FA(X)$.

2. (a) Let X be a finite set with $|X| = n$, and let $F = F(X)$ be the (standard) free group on X. Prove that $F/[F, F] \cong \mathbb{Z}^n$. **Hint:** By Problem 1 it is enough to show that $F/[F, F]$ is a free abelian group on X. Prove this using the fact that F is a free group on X and the following basic property of commutator subgroups (show why it is true): if $\phi : G \to A$ is a group homomorphism with A abelian, then $[G, G] \subseteq \operatorname{Ker} \phi$.
(b) Let X and Y be finite sets. Prove that free groups $F(X)$ and $F(Y)$ are isomorphic if and only if $|X| = |Y|$. **Hint:** The 'if' direction is easy. For the

'only if' direction use (a).

3. Let X be a set and $F(X)$ the free group on X. Given $f \in F(X)$, the *length of* f, denoted by $l(f)$ is defined to be the length of the unique reduced word in $X \cup X^{-1}$ representing f. Equivalently, $l(f)$ is the smallest n such that $f = x_1^{\epsilon_1} \ldots x_n^{\epsilon_n}$ for some $x_i \in X$ and $\epsilon_i \in \{\pm 1\}$.
(a) Prove that for any $f \in F(X)$ there exists integers $a, b \in \mathbb{Z}_{\geq 0}$ (depending on f) such that $l(f^n) = na + 2b$ for any $n \in \mathbb{N}$. Describe explicitly (i.e. give an algoirthm) how to compute a and b for a given f.
(b) Use (a) to show that free groups are torsion-free.

4. (a) Explain why for any $n \in \mathbb{N}$ there are only finitely many isomorphism classes of groups of order n.
(b) Let G be a finitely generated group and H a finite group. Prove that there are only finitely many homomorphisms from G to H. **Hint:** A homomorphism from G is completely determined by its values on generators.
(c) Let G be a finitely generated group. Prove that for any $n \in \mathbb{N}$ there are only finitely many normal subgroups of index n in G. Then deduce that G has only finitely many subgroups of index n (use small index lemma).

5. A group G is called *residually finite* if for any distinct elements $x, y \in G$ there exists a finite group H and a homomorphism $\phi : G \to H$ such that $\phi(x) \neq \phi(y)$. Thus, informally speaking, a group is residually finite if its elements can be separated via their images in finite quotients of G.
Clearly, any finite group G is residually finite (in which case for any $x, y \in G$ we take $H = G$ and ϕ the identity mapping). Probably, the simplest example of an infinite residually finite group is $\mathbb{Z}$: if $x, y \in \mathbb{Z}$ are distinct, choose any $n \in \mathbb{N}$ such that $n \nmid (x - y)$, and let $\phi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the natural projection; then clearly $\phi(x) \neq \phi(y)$.
(a) Prove that the following conditions on a group G are equivalent:

(i) G is residually finite

(ii) For any non-identity element $x \in G$ there exists a finite group H and a homomorphism $\phi : G \to H$ such that $\phi(x) \neq 1$.

(iii) The intersection of all normal subgroups of finite index in G is trivial.

(b) Prove that a subgroup of a residually finite group is residually finite.
(c) Let $n \geq 2$ be an integer. Prove that the group $GL_n(\mathbb{Z})$ is residually finite. **Hint:** Probably, it is the easiest to show that condition (ii) from part (a) holds.

Note: It is well known that free groups of finite rank can be embedded in $SL_2(\mathbb{Z})$. Thus, (b) and (c) imply that free groups of finite rank are residually finite (but there are other interesting proofs of that fact).

6. (optional) Recall from Homework#3 that a group G is called hopfian if any surjective homomorphism $\phi : G \to G$ is injective. Prove that any finitely generated residually finite group is hopfian. **Hint:** Suppose, on the contrary, that there exists a finitely generated group G and a surjective homomorphism $\phi : G \to G$ with $\operatorname{Ker}\phi$ non-trivial. Then $G/\operatorname{Ker}\phi \cong G$. Deduce that for any $n \in \mathbb{N}$ there is a bijection between the set of all normal subgroups of index n in G and the set of those normal sugbroups of index n in G which contain $\operatorname{Ker}\phi$. Then use Problem 4 to reach a contradiction with the assumption that G is residally finite.