Homework #7.

Plan for next week: Free groups (Section 6.3).

Problems, to be submitted by Thursday, October, 22nd

1. Let $n \geq 3$ and $G = D_{2n}$, the dihedral group of order $2n$.
(a) Prove that G contains a subgroup isomorphic to D_{2k} for any $k \mid n$.
(b) Prove that the dihedral group G is nilpotent if and only if n is power of 2.

2. (a) Let R be an associative ring with 1, and let $a, b \in R$ be such that $1+a$ and $1+b$ are invertible. Prove the following formula

$$(1+a)^{-1}(1+b)^{-1}(1+a)(1+b) = 1 + (1+a)^{-1}(1+b)^{-1}(ab - ba).$$

(b) Let R be an associative ring with 1 and $n \geq 2$ be an integer, and let $U_n(R)$ be the upper unitriangular subgroup of $GL_n(R)$. Prove that $U_n(R)$ is niplotent of class $n-1$ (we briefly outlined the proof in class). Note that you will need to apply (a) not to R itself but to the ring of $n \times n$ matrices over R.

Note: When you present your solution in (b), you may skip some computations, but the main points should be clear.

3. (a) Let $n \in \mathbb{N}$ be an integer, and suppose that for every non-prime divisor m of n there are no simple groups of order m. Prove that any group of order n is solvable.
(b) Prove that any group of order $p^k q$, where $p > q$ are distinct primes, is solvable.

4. Problems 31 and 32 on page 200 of DF. **Note:** Problem 31 follows very easily from what we proved in class a couple of weeks ago.