Homework #6.

Plan for next week: Nilpotent and solvable groups (the best approximation in Dummit and Foote is 6.1, but we will not follow it very closely).

Problems, to be submitted by Thursday, October, 15th

1. (a) Classify all abelian groups of order $360 = 2^3 \cdot 3^2 \cdot 5$ up to isomorphism. For each isomorphism type, state the corresponding elementary divisors form and invariant factors form.
(b) Let $n \in \mathbb{N}$, and decompose n as a product of primes: $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. Find the number of non-isomorphic abelian groups of order n. Express your answer in terms of the partition function.

2. Let G be a finite abelian group. Prove that G is cyclic if and only if G does not contain a subgroup isomorphic to $B \oplus B$ for some non-trivial group B.

3. (a) Let m and n be positive integers. What is the number of elements in $\mathbb{Z}/n\mathbb{Z}$ whose order divides m?
(b) Let G and H be finite abelian groups, and assume that for any $m \in \mathbb{N}$ the groups G and H have the same number of elements of order m. Prove that G and H are isomorphic. **Hint:** Let $a_m(G)$ denote the number of elements of order m in G. Use (a) to express the invariants factors of G in terms of $a_m(G)$ (the invariant factors of G are the integers appearing in the 'Invariant factors form' of the classification theorem)

4. Let G be an abelian group (not necessarily finitely generated), and let $Tor(G)$ be the set of elements of finite order in G. Recall that $Tor(G)$ is a subgroup of G (since G is abelian), called the torsion subgroup of G.
(a) Prove that the quotient group $G/Tor(G)$ is torsion-free.
(b) For each prime p let $Tor_p(G)$ be the set of elements of order p^k (with $k \geq 0$) in G. Prove that each $Tor_p(G)$ is a subgroup of $Tor(G)$ and that $Tor(G)$ is a direct sum of these subgroups where p runs over all primes.

5. Let Ω be a countable set (for simplicity you may assume that $\Omega = \mathbb{Z}$, the integers). Let $S(\Omega)$ be the group of all permutations of Ω. A permutation $\sigma \in S(\Omega)$ is called *finitary* if it moves only a finite number of points, that is, the set $\{i \in \Omega : \sigma(i) \neq i\}$ is finite. It is easy to see that finitary permutations form a subgroup of $S(\Omega)$ which will be denoted by $S_{fin}(\Omega)$. Finally, let

$A_{fin}(\Omega)$ be the subgroup of even permutations in $S_{fin}(\Omega)$ (note that it makes sense to talk about even permutations in $S_{fin}(\Omega)$, but not in $S(\Omega)$).
(a) Prove that the group $A_{fin}(\Omega)$ is simple and that $A_{fin}(\Omega)$ is a subgroup of index two in $S_{fin}(\Omega)$. **Hint:** To prove the first assertion solve problem 5 in [DF, page 151].
(b) Prove that $A_{fin}(\Omega)$ and $S_{fin}(\Omega)$ are both normal in $S(\Omega)$.
(c) Prove that neither of the groups $S(\Omega)$ and $S_{fin}(\Omega)$ is finitely generated. **Hint:** The two groups are not finitely generated for completely different reasons.
(d) Construct a finitely generated subgroup G of $S(\Omega)$ which contains $S_{fin}(\Omega)$. **Note:** This example shows that a subgroup of a finitely generated group does not have to be finitely generated.