Homework #4.

Approximate plan for next week: Basic applications of Sylow theorems (4.5, pp.142-146); Recognizing direct and semi-direct products (5.4, 5.5).

Problems, to be submitted by Thursday, September 24th

1. Let G be a group such that $G/Z(G)$ is cyclic. Prove that G is abelian.

2. Let G be a finite group and P a Sylow p-subgroup of G for some p.
(a) Let H be any subgroup of G. Prove that there exists $g \in G$ such that $H \cap gPg^{-1}$ is a Sylow p-subgroup of H. **Hint:** Start by choosing some Sylow p-subgroup of H and calling it, say, Q.
(b) Let H a subgroup of G such that $N_G(P) \subseteq H \subseteq G$. Prove that $N_H(P) = N_G(P)$ and $[G : H] \equiv 1 \mod p$.

3. Let $G = GL_n(\mathbb{Z}_p)$ and $P = U_n(\mathbb{Z}_p)$, the subgroup of upper-unitriangular matrices in $GL_n(\mathbb{Z}_p)$.
(a) Prove that P is a Sylow p-subgroup of G (you may use the formula for $|G|$ established in Homework#2).
(b) Let $B = UT_n(\mathbb{Z}_p)$, the subgroup of upper-triangular matrices in G. Prove that $N_G(P) = B$ (note that the inclusion $N_G(P) \supseteq B$ holds by Problem 1 in Homework#3). **Hint:** Consider the natural action of G on $X = \mathbb{Z}_p^n$. First compute $X^P = \{x \in X : gx = x \text{ for any } g \in P\}$, the set of fixed points of P. Now if $g \in N_G(P)$, what can you say about $g \cdot X^P$ (the image of X^P under the action of g). Use this result and induction on n to prove that $N_G(P) \subseteq B$.
(c) Find $n_p(G)$, the number of Sylow p-subgroups of G.

Definition: A group G is called *simple* if it has no non-trivial proper normal subgroups (that is, the only normal subgroups of G are G itself and $\{1\}$).

4. (a) Prove that a group of order 200 is not simple.
(b) Prove that a group of order 72 is not simple.

5. Prove that a group of order 132 is not simple.