Homework #3, to be submitted by Thursday, September, 17th

1. (a) Let $(G, X, \cdot)$ be a group action. For a subset S of X we put $G_S = \{g \in G : g \cdot s = s \text{ for any } s \in S\}$ (the poinwise stabilizer of S) and $G_{\{S\}} = \{g \in G : g \cdot S = S\}$ (the stablizer of S).
Prove that G_S is a normal subgroup of $G_{\{S\}}$.
(b) Let R be a commutative ring with 1 and $n \geq 2$ an integer. Let $UT_n(R)$ be the subgroup of $GL_n(R)$ consisting of upper-triangular matrices and $U_n(R)$ the subgroup of $GL_n(R)$ consisting of upper-unitriangular matrices (upper-triangular matrices with 1's on the diagonal). Use (a) to prove that $U_n(R)$ is normal in $UT_n(R)$.

2. An action of a group G on a set X is called *transitive* if it has just one orbit, that is, for any $x, y \in X$ there exists $g \in G$ with $g \cdot x = y$.
(a) Let $(G, X, \cdot)$ be a group action. Prove that if $x, y \in X$ lie in the same orbit, then their stabilizers G_x and G_y are conjugate, that is, there exists $g \in G$ with $gG_xg^{-1} = G_y$.
(b) Suppose that $(G, X, \cdot)$ is a transitive action, and fix $x \in X$. Prove that the kernel of this action is equal to $\cap_{g \in G} gG_xg^{-1}$
(c) Now suppose that G and X are both finite and $(G, X, \cdot)$ is a transitive faithful action (where 'faithful' means the kernel is trivial) and G is abelian. Prove that for any $g \in G \setminus \{1\}$ the fixed set X^g is empty. Deduce that $|X| = |G|$. **Hint:** Use (b).

3. Let G be a group. For each $g \in G$ let $\iota_g : G \to G$ be the conjugation by g, that is, $\iota_g(x) = gxg^{-1}$. Recall that $\iota_g \in \mathrm{Aut}(G)$ for any $g \in G$ and the mapping $\iota : G \to \mathrm{Aut}(G)$ given by $\iota(g) = \iota_g$ is a homomorphism. Elements of the subgroup $\mathrm{Inn}(G) = \iota(G)$ of $\mathrm{Aut}(G)$ are called inner automorphisms.
(a) Prove that for any $g \in G$ and $\sigma \in \mathrm{Aut}(G)$ one has $\sigma\iota_g\sigma^{-1} = \iota_{\sigma(g)}$. Deduce that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
(b) Let H be a normal subgroup of G. Note that for each $g \in G$, the mapping ι_g restricted to H is an automorphism of H. By slight abuse of notation we denote this automorphism of H by ι_g as well. Prove that ι_g is an inner automorphism of H if and only if $g \in H \cdot C_G(H)$ where $C_G(H)$ is the centralizer of H in G.
(c) Use (b) to find a non-inner automorphism of A_n for $n \geq 3$.

4. Let $n \geq 4$ and $f = (1,2)(3,4) \in S_n$. Prove that $|C_{S_n}(f)| = 8(n-4)!$. Then describe elements of this centralizer explictly. **Hint:** What is the conjugacy class of f?

5. (optional) *Necklace-counting problem:* Suppose that we want to build a necklace using n beads of k possible colors (we do not have to use all available colors). Two necklaces will be considered equivalent if they can be obtained from each other using rotations or reflections. What is the number of non-equivalent necklaces one can construct?

Approach using group actions. Let X be the set of all necklaces with beads of k possible colors located at the vertices of a regular n-gon. The dihedral group D_{2n} has a natural action on X, and the orbits under that action are precisely equivalence classes of necklaces in the above sense. Use this interpretation and Burnside's lemma to prove that for $n = 9$ the number of non-equivalent necklaces is

$$\frac{k^9 + 2k^3 + 6k + 9k^5}{18}.$$

Then try to do the same for general n.

6. (a) Prove that if $\phi : \mathbb{Z} \to \mathbb{Z}$ is a surjective homomorphism, then ϕ must be injective.

(b) Find a group G and a homomorphism $\phi : G \to G$ which is surjective and not injective.

(c) (optional) A group G with the property that every surjective homomorphism from G to G must also be injective is called *hopfian*, so part (a) asserts that $\mathbb{Z}$ is hopfian. Can you find interesting sufficient conditions for a group to be hopfian?

7. (a) (practice) Let G be a group. Let $x, y \in G$ be such that $xy = yx$, x has order m, y has order k with $m, k < \infty$ and $gcd(m,k) = 1$. Prove that xy has order mk.

(b) Let G be an abelian group. Prove that the set of elements in G which have finite order is a subgroup of G. This subgroup is called the *torsion subgroup* of G.

(c) Give an example of a group where the set of elements of finite order is not a subgroup. **Note:** Obviously, the group has to be infinite and non-abelian by (a).