Homework #9. Due Tuesday, November 23rd

Reading:

1. For this homework assignment: online class notes (Lectures 18-21) and Steinberg, parts of 4.2-4.4.

2. Plan for upcoming classes. Thu, Nov 18: permutation representations (online Lecture 22, Chapter 7 in Steinberg). Tue, Nov 23: TBA.

Problems:

For problems (or their parts) marked with a *, a hint is given later in the assignment. Do not to look at the hint(s) until you seriously tried to solve the problem without it.

1. Let p be a prime and $G = \mathrm{Heis}(\mathbb{Z}_p)$, the Heisenberg group over $\mathbb{Z}_p$ defined in HW#7.2

(a) Determine the number of conjugacy classes of G and their sizes. As in HW#8.6, you can work directly with matrices or with their expressions in terms of the generators x, y, z introduced in HW#7.2.

(b) Let $\omega \neq 1$ be a p^{th} root of unity, that is, $\omega = e^{\frac{2\pi k i}{p}}$ with $1 \leq k \leq p-1$. Let V be a p-dimensional complex vector space with basis $e_{[0]}, e_{[1]}, \ldots, e_{[p-1]}$ where we think of indices as elements of $\mathbb{Z}_p$. Prove that there exists a representation (ρ_ω, V) of G such that

- $\rho_\omega(z)e_{[k]} = \omega e_{[k]}$ for each k (that is, $\rho_\omega(z)$ is just the scalar multiplication by ω),
- $\rho_\omega(y)e_{[k]} = e_{[k+1]}$ for each k (that is, $\rho_\omega(y)$ cyclically permutes the basis vectors) and finally
- $\rho_\omega(x)e_{[k]} = \omega^k e_{[k]}$ for each k.

(c) Prove that every representation in (b) is irreducible (do not do this directly from definition) and every irreducible complex representation of G is either one-dimensional or equivalent to (ρ_ω, V) for some ω.

2. Let (ρ, V) be a representation of a group G. Recall that the dual representation (ρ^*, V^*) is defined by $\rho^*(g)(f) = f \circ \rho(g)^{-1}$ for all $f \in V^*$. Prove parts (1) and (2) of Claim 21.1 from class:

(1) (ρ^*, V^*) is indeed a representation

(2) If β is any basis of V and β^* is the dual basis of V^*, then $[\rho^*(g)]_{\beta^*} = ([\rho(g)]_\beta^{-1})^T$

Note: Part (2) has almost nothing to do with representation theory. Recall from Lecture 9 that given an operator $A \in \mathrm{End}(V)$, its adjoint $A^* \in \mathrm{End}(V^*)$ is defined by $A^*(f) = f \circ A$ (recall that this notion of adjoint is related to but slightly different from adjoints in complex inner product spaces). Note that the homomorphism ρ^* from the definition of the dual representation can be expressed in terms of adjoints as follows:

$$\rho^*(g) = (\rho(g)^{-1})^* \text{ for all } g \in G.$$

What you really need to prove is Claim 9.2 from online notes which asserts that $[A^*]_{\beta^*} = ([A]_\beta)^T$ (make sure to explain how (2) follows from this).

3.* Let $G = S_n$ for some $n \geq 2$ and χ an irreducible complex character of G. Prove that χ is real-valued, that is, $\chi(g) \in \mathbb{R}$ for all $g \in G$.

4. Give an example of two representations V and W of the same group which are not equivalent, but have the same dimension and the same character. Recall that by Corollary 22.2 from class this cannot happen if G is finite and representations are complex.

5. Let (ρ, V) and (ρ', V) be complex representations of a finite group G (the vector space V is the same for both representations). Suppose that $\rho'(g)$ is conjugate to $\rho(g)$ in $\mathrm{GL}(V)$ for every $g \in G$. Prove that the representations (ρ, V) and (ρ', V) are equivalent. **Note:** The result is not automatic since the matrix which conjugates $\rho'(g)$ to $\rho(g)$ may depend on g.

Hint for 3: Use our discussion in Lecture 21 (=online Lecture 18) to find a general condition on a finite group G which guarantees that all of its complex characters are real-valued and then show that this condition holds for S_n.