Homework #6. Due Saturday, October 23rd

Reading:

1. For this homework assignment: online class notes (Lectures 12 and 13) and Steinberg 3.2, 4.1
2. Next week we will talk about Schur's Lemma and representations of abelian groups and one-dimensional representations (online lectures 13 and 14 and Steinberg 4.1).

Problems:

1. The goal of this problem is to prove the general case of Maschke's Theorem: *If G is a finite group and F is any field with* $\mathrm{char}(F) \nmid |G|$, *then any representation of G over F is completely reducible.*

The key part of the proof is the following lemma.

Lemma: *Let G and F be as above, (ρ, V) a representation of G over F and W a G-invariant subspace. Then there exists a G-invariant subspace U such that $V = W \oplus U$.*

Maschke's theorem follows immediately by repeated applications of this lemma (or by induction on $\dim V$).

To prove the lemma, consider the following maps. Choose any subspace Z of V such that $V = W \oplus Z$ and let $P : V \to W$ be the projection onto W along Z, that is, P is the unique linear map such that $P(w) = w$ for all $w \in W$ and $P(Z) = 0$. Now define $Q : V \to V$ by

$$Q(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)^{-1} P \rho(g)(v).$$

Prove that

- (i) $Q(w) = w$ for all $w \in W$ and $\mathrm{Im}(Q) = W$. Deduce that $Q^2 = Q$.
- (ii) $\mathrm{Ker}(Q)$ is G-invariant

Deduce from (i) that $V = W \oplus \mathrm{Ker}(Q)$ and therefore $U = \mathrm{Ker}(Q)$ has the desired property.

2. Given a vector space V, let $\mathrm{End}(V) = \mathrm{Hom}(V, V)$ (we previously denoted this set by $\mathcal{L}(V)$). Elements of $\mathrm{End}(V)$ are called *endomorphisms of V*. If X is an algebraic structure (e.g. group, ring, vector space), an endomorphism of X is a homomorphism from X to itself.

(a) Prove that $\mathrm{End}(V)$ is a ring with 1 (where addition is the usual pointwise addition of maps and multiplication is given by composition). Clearly state where you use the fact that elements of $\mathrm{End}(V)$ are linear maps.

Now suppose that (ρ, V) is a representation of some group G. Let $\mathrm{End}_\rho(V)$ be the set of those elements of $\mathrm{End}(V)$ which are homomorphisms of representations (from (ρ, V) to (ρ, V)). Prove that

(b) $\mathrm{End}_\rho(V)$ is a subring of $\mathrm{End}(V)$ which contains 1 and also that $\mathrm{End}_\rho(V)$ is a vector subspace of $\mathrm{End}(V)$.

(c) If $g \in G$ is a central element (that is, $gx = xg$ for all $x \in G$), then $\rho(g) \in \mathrm{End}_\rho(V)$.

3. Next week in class we will show that if G is an abelian group, then any irreducible representation of G over an algebraically closed field is one-dimensional.

(a) Let $n > 2$ be an integer. Construct an irreducible representation of $\mathbb{Z}_n$ over $\mathbb{R}$ (reals) which is not one-dimensional. **Hint:** Recall that we completely described representations of cyclic groups over arbitrary fields in Lecture 14 on Oct 7.

(b) Prove that any irreducible representation of $\mathbb{Z}_2$ over any field is one-dimensional.

(c) (bonus) Describe (with proof) all finite abelian groups G such that any irreducible representation of G over any field is one-dimensional.

4. Let (α, V) and (β, W) be representations of the same group over the same field. The *tensor product* of these representations is the representation $(\rho, V \otimes W)$ where $\rho(g) = \alpha(g) \otimes \beta(g)$ for each $g \in G$ (in the notations from Problem 2 in HW#5). In other words, $\rho(g) \in GL(V \otimes W)$ is the unique linear map such that $\rho(g)(v \otimes w) = (\alpha(g)v) \otimes (\beta(g)w)$ for all $v \in V$ and $w \in W$.

(a) Prove that $(\rho, V \otimes W)$ is indeed a representation, that is, $\rho : G \to GL(V \otimes W)$ is a homomorphism.

(b) Prove that if (α, V) is NOT irreducible and $W \neq 0$, then $(\rho, V \otimes W)$ is not irreducible either.

(c) Now prove that if (α, V) is irreducible and $\dim(W) = 1$, then $(\rho, V \otimes W)$ is irreducible.