Homework #4. Due Saturday, Sep 25

1. Reading for this homework assignment: Friedberg-Insel-Spence 6.1, 6.3 + online class notes (Lectures 7,8)

2. Plan for the next week: Finish diagonalization in Inner Product Spaces (Friedberg-Insel-Spence 6.4, 6.5, online lectures 7,8), dual spaces (Friedberg-Insel-Spence 2.6, online lectures 9,10); start talking about tensor products (online lecture 10).

Problems:

For problems (or their parts) marked with a *, a hint is given later in the assignment. Do not to look at the hint(s) until you seriously tried to solve the problem without it.

Note: Problems 1, 2 and 3 establish some fundamental facts about unitary operators that we will continuously use when talking and representations.

1. Let V be a finite-dimensional inner product space over $\mathbb{C}$ and let $A \in \mathcal{L}(V)$ be a normal operator. Prove that A is unitary if and only if all eigenvalues of A have absolute value 1.

2. Let V be an inner product space over $\mathbb{C}$ and $A \in GL(V)$, that is, $A \in \mathcal{L}(V)$ is invertible.

- (a) Prove that A is unitary if and only if $\langle Ax, Ay \rangle = \langle x, y \rangle$ for all $x, y \in V$.
- (b)* Now use (a) to prove that A is unitary if and only if $\|Ax\| = \|x\|$ for all $x \in V$.

3. Let V be an inner product space over $\mathbb{C}$, let $A \in \mathcal{L}(V)$ be unitary, and let $W \subseteq V$ be a *finite-dimensional* subspace of V which is A-invariant (that is, $A(W) \subseteq W$).

- (a) Prove that if $A(W) = W$.
- (b) Use (a) to prove that $W^{\perp}$ is also A-invariant.

4. Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Find a unitary matrix U such that $U^{-1}AU$ is diagonal (see the online version of Lecture 8 for a brief discussion of the algorithm for finding U). Try to determine the eigenvalues of A without computing the characteristic polynomial.

5. Let V be a finite-dimensional inner product space over $\mathbb{C}$ and let H and G be Hermitian forms on V.

(a)* Assume that G is positive-definite. Prove that there exists a basis β of V such that $[H]_\beta$ and $[G]_\beta$ are both diagonal (equivalently, if $A, B \in Mat_n(\mathbb{C})$ are Hermitian matrices and A is positive definite, there exists $P \in GL_n(\mathbb{C})$ such that P^*AP and P^*BP are both diagonal).

(b) (bonus) Now give an explicit example showing that if neither G nor H is positive-definite, the conclusion of (a) may fail.

Hint for 2(b): One direction is straightforward. For the other direction use the result of an earlier homework problem.

Hint for 5: The result follows from one of the theorems from the online lecture 8 with almost no additional computations (but it may take some work to figure out which result to use).