**Homework #3. Due Saturday, Sep 18**

1. Reading for this homework assignment: Friedberg-Insel-Spence 6.1, 6.3 + online class notes (Lectures 5,6)

2. Plan for the next week: Diagonalization in Inner Product Spaces (Friedberg-Insel-Spence 6.4, 6.5, online lectures 7,8)

**Problems:**

**1.** Let $V$ be a finite-dimensional vector space over a field $F$ of characteristic 2 and $H$ a symmetric (=skew-symmetric since char$F = 2$) bilinear form on $V$. Prove that there exist subspaces $V_1$ and $V_2$ of $V$ such that

(a) $V = V_1 \oplus V_2$ and $V_1 \perp V_2$ (that is, $H(v, w) = 0$ for all $v \in V_1$ and $w \in V_2$).
(b) $H_{|V_1}$ is diagonalizable (that is, $[H_{|V_1}]_{\beta_1}$ is diagonal for some basis $\beta_1$ of $V_1$)
(c) $H_{|V_2}$ is alternating and non-degenerate (such a form is called symplectic).

**Hint:** Combine the proofs of Theorems 4.2 and 5.1 from class.

**2.** As observed at the beginning of Lecture 5, if $V$ is a finite-dimensional vector space and $H$ is a bilinear form on $V$, the following conditions are equivalent:

(i) $H$ is left non-degenerate
(ii) $H$ is right non-degenerate
(iii) $[H]_\beta$ is invertible for some (hence any) basis $\beta$ of $V$.

Now assume that $\dim V$ is countably infinite and $\beta = \{v_1, v_2, \ldots\}$ is a basis of $V$.

(a) Find (with proof) conditions $(ND)_{left}$ and $(ND)_{right}$ on an infinite square matrix such that $H$ is left non-degenerate $\iff$ $[H]_\beta$ satisfies $(ND)_{left}$ and $H$ is right non-degenerate $\iff$ $[H]_\beta$ satisfies $(ND)_{right}$. **Hint:** The conditions will be non-equivalent, but in the finite dimensional case they should both reduce to different (but similar) well-known characterizations of invertible matrices.
(b) Use (a) to find a bilinear form $H$ on $V$ which is left-nondenerate but not right-nondegenerate.

**3.** Let $V$ be an inner product space.

(a) Prove the parallelogram law: $\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2)$ for all $x, y \in V$.

(b) Show that $\langle x, y\rangle$ can be expressed as a linear combination of squares of norms. In Lecture 6 we discussed how to do this for the real inner product spaces.

**4.** Let $V$ be a finite-dimensional complex inner product space and $A \in \mathcal{L}(V)$. Prove that $\mathrm{Im}(A^*) = \mathrm{Ker}(A)^\perp$ (where the orthogonal complement is with respect to the inner product on $V$). Here $A^*$ is the adjoint operator of $A$ (see the end of the online lecture 6 for the definition).

**5.** Let $V$ be an inner product space where $\dim V$ is finite or countable, $\beta$ an orthonormal basis of $V$ and $A \in \mathcal{L}(V)$.

(a) Prove that if $A^* \in \mathcal{L}(V)$ is any operator such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in V$, then $[A^*]_\beta = [A]_\beta^*$ (where $[A]_\beta^*$ is the conjugate transpose of $A$). In particular, this shows that the adjoint operator is unique (if exists).

(b) As we will prove in class, the adjoint $A^*$ always exists if $\dim V$ is finite. Now use (a) and a result from earlier homeworks to show that if $V$ is countably-dimensional, then the adjoint $A^*$ may not exist.