Homework #1. Due by 6pm on Saturday, Sep 4th

Problems

Note on hints: All hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

Most of the problems below deal with concepts that have not been introduced in class so far. The definitions of those concepts are given on page 3. We denote by $Mat_n(F)$ the set of all $n \times n$ matrices over F.

1. Let $V = Pol_2(\mathbb{R})$, the vector space of polynomials of degree at most 2 over $\mathbb{R}$. Let $\beta = \{1, x, x^2\}$ and $\gamma = \{1, (x-1), (x-1)^2\}$. Both β and γ are bases of V (you do not need to verify this). Let $T : V \to V$ be the differentiation map: $T(f) = f'$.

(a) compute the matrix $[T]_\beta$ directly from definition
(b) compute the matrix $[T]_\gamma$ directly from definition
(c) now compute $[T]_\gamma$ using your answer in (a) and the change of basis formula.

2. In each of the following examples determine if H is a bilinear form on V (make sure to justify your answer):

(a) $V = Mat_n(F)$ for some field F and $n \in \mathbb{N}$ and $H(A, B) = AB$.
(b) $V = Mat_n(F)$ for some field F and $n \in \mathbb{N}$ and $H(A, B) = (AB)_{1,1}$ (the (1,1)-entry of the matrix AB).
(c) $V = F^n$ for some field F and $n \in \mathbb{N}$ and $H((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = x_1 + y_1$.

3. As in problem 1, let $V = Pol_2(\mathbb{R})$, and define $H : V \times V \to \mathbb{R}$ by

$$H(f, g) = \int_0^1 f(x)g(x)dx.$$

Prove that H is a symmetric bilinear form and compute the matrix $[H]_\beta$ (where again $\beta = \{1, x, x^2\}$).

4. Let F be any field, $n \in \mathbb{N}$ and $V = Mat_n(F)$, the vector space of $n \times n$ matrices over F. Let e_{ij} be the matrix whose (i, j)-entry is equal to 1 and all other entries are 0. Then $\beta = \{e_{ij} : 1 \leq i, j \leq n\}$ is a basis of V (you do not need to verify this). Define $H : V \times V \to F$ by

$$H(A, B) = Tr(AB^T)$$

(where B^T is the transpose of B). Prove that H is a symmetric bilinear form and compute the matrix $[H]_\beta$ (you can order β in any way you like). Include all the relevant computations.

5. Let F be a field with char$(F) \neq 2$, let V be a finite-dimensional vector space over F, and let H be a bilinear form on V. Prove that H can be **uniquely** written as $H = H^+ + H^-$ where H^+ is a symmetric bilinear form on V and H^- is an antisymmetric bilinear form on V.

6. Let F be any field and $n \in \mathbb{N}$.

(a) Let $V = F^n$ (the standard n-dimensional vector space over F). Let $D : V \times V \to F$ be the dot product form. Prove that D is non-degenerate.

(b)* Now V be any n-dimensional vector space over F, β an ordered basis for V and H a bilinear form on V. Prove that H is left non-degenerate if and only if $[H]_\beta$ (the matrix of H with respect to β) is invertible.

Note: (a) is a special case of (b); however, there is a natural way to solve (b) using (a), so it does make sense to prove (a) first.

7. Let F be any field, $n \in \mathbb{N}$, $V = F^n$ and $\{e_1, \ldots, e_n\}$ the standard basis of V. Define $\rho : S_n \to GL(V)$ by $(\rho(g))(e_i) = e_{g(i)}$. As discussed in Lecture 1, the pair (ρ, V) is a representation of S_n.

(a) Let V_0 be the subspace of V consisting of all vectors whose sum of coordinates is equal to 0:

$$V_0 = \{(x_1, \ldots, x_n) \in V : x_1 + \ldots + x_n = 0\}.$$

Prove that V_0 is an S_n-invariant subspace of V, and therefore (ρ, V_0) is also a representation of S_n.

(b)* **BONUS** Now prove that the representation (ρ, V_0) is irreducible, that is, if W is any S_n-invariant subspace of V_0, then $W = 0$ or $W = V_0$.

Definitions

1. *Characteristic of a ring.* Let R be a ring with 1. The *characteristic* of R, denoted $\text{char}(R)$, is the smallest positive integer n such that $\underbrace{1 + \ldots + 1}_{n \text{ times}} = 0$ in R. If no such n exists, we define $\text{char}(R) = 0$. For instance, $\text{char}(\mathbb{Z}) = \text{char}(\mathbb{Q}) = \text{char}(\mathbb{R}) = \text{char}(\mathbb{C}) = 0$, while $\text{char}(\mathbb{Z}_n) = n$ (where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the ring of congruence classes mod n). There is a theorem saying that if F is a field, then $\text{char}(F)$ is either 0 or a prime number.

2. Let V be a vector space over any field. A *bilinear form* on V is a map $H : V \times V \to F$ which is linear in each variable, that is,

$$H(x+\lambda y, z) = H(x, z)+\lambda H(y, z) \text{ and } H(x, y+\lambda z) = H(x, y)+\lambda H(x, z)$$

for all $\lambda \in F, x, y, z \in V$.
If V is finite-dimensional and $\beta = \{v_1, \ldots, v_n\}$ is an ordered basis of V, the matrix of H with respect to β, denoted by $[H]_\beta$ is the $n \times n$ matrix over F whose (i, j)-entry is $H(v_i, v_j)$.

3. Let H be a bilinear form on a vector space V. Then H is called

   (i) *symmetric* if $H(x, y) = H(y, x)$ for all $x, y \in V$;

   (ii) *antisymmetric* if $H(x, y) = -H(y, x)$ for all $x, y \in V$;

   (iii) *left non-degenerate* if for every nonzero $x \in V$ there exists $y \in V$ with $H(x, y) \neq 0$.

   (iv) *right non-degenerate* if for every nonzero $x \in V$ there exists $y \in V$ with $H(y, x) \neq 0$.

Hint for 6(b). Use the formula $H(v, w) = [v]_\beta^T [H]_\beta [w]_\beta$ (will be proved in Lecture 3). Interpret the right-hand side of this formula as a dot product and use 6(a).

Hint for 7(b). Let W be an S_n-invariant subspace of V_0, and assume that $W \neq 0$. Our goal is to show that $W = V_0$. First prove that W must contain a nonzero vector w_1 one of whose coordinates is zero. Then use w_1 to construct an element $w_2 \in W$ which has exactly two nonzero coordinates and deduce that w_1 is a nonzero scalar multiple of $e_i - e_j$ for some $i \neq j$. Finally, use w_2 to prove that $W = V_0$.