**Homework #10. Due on Tuesday, April 29th**

**Reading:**

1. For this homework assignment: class notes on Pell's equation and Chapter 11, Sections 11.1-11.5. For more information on Pell's equation see `http://math.stanford.edu/~jbooher/expos/continued_fractions.pdf`

**Some terminology and notations on continued fractions:**

Let $a_0, a_1, a_2, \ldots$ be a finite or infinite sequence of real numbers satisfying $a_n \geq 1$ for $n > 0$. The associated continued fraction $a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}$ is denoted by $[a_0; a_1, a_2, \ldots]$. In the case of a finite continued fraction $[a_0; a_1, \ldots, a_n]$, the number $n$ will be called the *height* of the continued fraction.

From now on assume that the sequence $\{a_n\}$ is infinite. The numbers $C_0 = [a_0]$, $C_1 = [a_0; a_1], C_2 = [a_0; a_1, a_2], \ldots$ are called the *convergents* of the continued fraction $[a_0; a_1, \ldots]$. Define two sequences $\{A_n\}_{n \geq -1}$ and $\{B_n\}_{n \geq -1}$ by $A_{-1} = 1$, $B_{-1} = 0$, $A_0 = a_0$, $B_0 = 1$ and

$$A_{n+1} = a_{n+1} A_n + A_{n-1} \text{ and } B_{n+1} = a_{n+1} B_n + B_{n-1} \text{ for } n \geq 0.$$

In class we proved that $C_n = \frac{A_n}{B_n}$ for all $n \geq 0$.

**Problems:**

1. Let $\alpha \in \mathbb{R}$, and assume that the continued fraction for $\alpha$ is infinite periodic. Prove that $\alpha$ is a quadratic irrational, that is, $\alpha \notin \mathbb{Q}$, but $\alpha$ is a root of a nonzero quadratic polynomial with integer coefficients. **Hint:** Start with the case when the continued fraction for $\alpha$ is purely periodic, that is, the periodic part starts from the very beginning ($\alpha = [\overline{a_0, \ldots, a_{k-1}}]$). Start by writing down some equation that $\alpha$ must satisfy (it will involve a finite continued fraction) and then conclude that $\alpha$ satisfies a quadratic equation. Then use the result in the purely periodic case to establish the general case.

2 (optional). Consider an infinite continued fraction $[a_0; a_1, a_2, \ldots]$, and let $A_n, B_n$ and $C_n$ be defined as above.

(i) Prove that $A_{n-1}B_n - B_{n-1}A_n = (-1)^n$ for all $n \geq 0$.

(ii) Prove that $B_n \geq B_{n-1}$ for all $n \geq 0$ and $B_n \geq 2B_{n-2}$ for all $n \geq 2$.

(iii) Prove that $|C_n - C_{n+1}| \leq \frac{1}{2^n}$ for all $n \geq 0$.

(iv) Deduce from (iii) that the sequence $\{C_n\}$ is Cauchy and therefore converges.

3. Find a non-trivial solution to Pell's equation $x^2 - dy^2 = 1$ in each of the following cases:

(i) $d = (a^2 - 1)$ for some $a \in \mathbb{N}$

(ii) $d = a^2 + 1$ for some $a \in \mathbb{N}$

(iii) $d = a(a + 1)$ for some $a \in \mathbb{N}$

**Hint:** In case (i) a solution is easy to guess; in cases (ii) and (iii) one can also guess; alternatively you can use continued fractions.

4. Use continued fractions to find a solution to Pell's equation $x^2 - dy^2 = 1$ for $d = 19$ and $d = 41$.

5. Prove that for every $n \in \mathbb{N}$ there exists a solution to the equation $x^2 - 3y^2 = 1$ satisfying $10^n < x < 10^{n+1}$. **Hint:** How are all solutions obtained from the fundamental solution?

6. Let $(x, y, z)$ be a primitive integer solution for the equation $x^2 + 2y^2 = z^2$. Prove that there exist integers $u$ and $v$ such that $(x, y, z) = (2u^2 - v^2, 2uv, 2u^2 + v^2)$ or $(u^2 - 2v^2, 2uv, u^2 + 2v^2)$.