Math 5653. Number Theory. Spring 2013. First Midterm.
Wednesday, February 27th, 2-3:20pm

Directions: No books, notes, calculators, laptops, PDAs, cellphones, web appliances, or similar aids are allowed. All work must be your individual efforts.

- Show all your work and justify all statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

problem	1	2	3	4	5	total	bonus	overall
points	10	10	10	10	10	40	5	45
score								

Scoring system: Exam consists of **5** problems, each of which is worth **10** points. Your regular total is the sum of the best **4** out of **5** scores (so the maximum regular total is 40). If k is the lowest of your 5 scores and $k > 5$, you will get $k - 5$ bonus points (so the maximum total with the bonus is 45).

Problem 1: Find the SMALLEST positive integer x satisfying each of the following congruences:

$$x \equiv 2 \mod 3; \qquad x \equiv 3 \mod 5; \qquad x \equiv 5 \mod 7$$

You can use any of the methods we discussed, but blind guessing will not be accepted. In any case, you need to justify why x you found is the smallest positive solution.

Problem 2: Prove that the equation

$$x_1^{11} + x_2^{11} + \ldots + x_{10}^{11} = 230000000000011$$

has no integer solutions. **Hint:** reduce modulo a suitable prime.

Problem 3: Find the smallest positive integer m such that

$$x^m \equiv 1 \mod 120 \text{ for all } x \text{ which are coprime to } 120.$$

Note that $120 = 3 \cdot 5 \cdot 8$.

Problem 4: Let p be an odd prime and a an integer not divisible by p. Find the number of solutions mod p^3 to the following congruence

$$x^3 - a^2x^2 + p^2 \equiv 0 \mod p^3.$$

Hint: It is possible (and actually not hard) to find some of these solutions explicitly, but not all of them.

Problem 5: Let n be a positive integer. Prove that $\sqrt{n}$ is always either an integer or an irrational number.

Hint: Let $n = p_1^{a_1} \dots p_k^{a_k}$ be a prime factorization of n. Start by guessing a simple condition (C) on the sequence of exponents $(a_1, \dots, a_k)$ such that $\sqrt{n}$ is an integer if (C) holds and $\sqrt{n}$ is irrational if (C) fails; then prove that your guess is correct. Some points will be given just for a correct guess.

Extra page 1.

Extra page 2.