Homework #6. Due Wednesday, March 6th, in class

Reading:

1. For this homework assignment: Chapter 6.
2. For the next two classes: beginning of Chapter 7.

Problems:

1.

(a) Let G be a cyclic group of order n, and let m be a positive integer. Let $P_m(G)$ be the set of all elements of G which can be written as g^m for some $g \in G$. Prove that $|P_m(G)| = \frac{n}{gcd(n,m)}$

(b) Let p and q be distinct odd primes. Prove that the number of mod p solutions to the congruence $x^q \equiv 1 \mod p$ is equal to $gcd(q, p-1)$.

2. Deduce directly from Theorem 6.10 in the book that the group U_{2^a}, with $a \geq 3$, contains precisely four elements g satisfying $g^2 = e$ and find those elements explicitly.

3. Let $G_1, \ldots, G_k$ be finite groups.

(a) Prove that if each G_i is abelian, then

$$o_{max}(G_1 \times \ldots \times G_k) = lcm(o_{max}(G_1), \ldots, o_{max}(G_k)),$$

where as before $o_{max}(G) = \max\{o(g) : g \in G\}$. State clearly how you use that G_i are abelian.

(b) Give an example showing that assertion of (a) maybe false without the assumption that G_i are abelian.

4. Prove that the equation

$$x_1^{11} + x_2^{11} + \ldots + x_{10}^{11} = 230000000000011$$

has no integer solutions. **Hint:** reduce modulo a suitable prime.

5. Find the smallest positive integer m such that

$$x^m \equiv 1 \mod 120 \text{ for all } x \text{ which are coprime to } 120.$$

Note that $120 = 3 \cdot 5 \cdot 8$.

6. Let p be an odd prime and a a (fixed) integer not divisible by p. Find the number of solutions mod p^3 to the following congruence

$$x^3 - a^2x^2 + p^2 \equiv 0 \mod p^3.$$