Homework #11. Due on Tuesday, April 30th

Reading:

1. For this homework assignment: class notes on Pell's equation and Chapter 11, Sections 11.1-11.4.

Some terminology and notations on continued fractions:

Let $a_0, a_1, a_2, \ldots$ be a finite or infinite sequence of real numbers satisfying $a_n \geq 1$ for $n > 0$. The associated continued fraction $a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}$ is denoted by $[a_0; a_1, a_2, \ldots]$. In the case of a finite continued fraction $[a_0; a_1, \ldots, a_n]$, the number n will be called the *height* of the continued fraction.

From now on assume that the sequence $\{a_n\}$ is infinite. The numbers $C_0 = [a_0]$, $C_1 = [a_0; a_1], C_2 = [a_0; a_1, a_2], \ldots$ are called the *convergents* of the continued fraction $[a_0; a_1, \ldots]$. Define two sequences $\{A_n\}_{n \geq -1}$ and $\{B_n\}_{n \geq -1}$ by $A_{-1} = 1$, $B_{-1} = 0$, $A_0 = a_0$, $B_0 = 1$ and

$$A_{n+1} = a_{n+1}A_n + A_{n-1} \text{ and } B_{n+1} = a_{n+1}B_n + B_{n-1} \text{ for } n \geq 0.$$

In class we proved that $C_n = \frac{A_n}{B_n}$ for all $n \geq 0$.

Problems:

1. Prove that a real number α cannot be represented by two distinct integer continued fractions, if we require that for a finite continued fraction of positive height, the last entry must be different from 1. **Hint:** Assume that $[a_0; a_1, \ldots,] = [b_0; b_1, \ldots,]$ are two integer continued fractions (satisfying the above restriction) representing the same real number. First show that $a_0 = b_0$ and then proceed inductively.

2. Let $\alpha \in \mathbb{R}$, and assume that the continued fraction for α is infinite periodic. Prove that α is a quadratic irrational, that is, $\alpha \notin \mathbb{Q}$, but α is a root of a nonzero quadratic polynomial with integer coefficients. **Hint:** Start with the case when the continued fraction for α is purely periodic, that is, the periodic part starts from the very beginning ($\alpha = [\overline{a_0, \ldots, a_{k-1}}]$). Start by writing down some equation that α must satisfy (it will involve a finite continued fraction) and then conclude that α satisfies a quadratic equation. Then use the result in the purely periodic case to establish the general case.

3. Consider an infinite continued fraction $[a_0; a_1, a_2, \ldots]$, and let A_n, B_n and C_n be defined as above.

(i) Prove that $A_{n-1}B_n - B_{n-1}A_n = (-1)^n$ for all $n \geq 0$.

(ii) Prove that $B_n \geq B_{n-1}$ for all $n \geq 0$ and $B_n \geq 2B_{n-2}$ for all $n \geq 2$.

(iii) Prove that $|C_n - C_{n+1}| \leq \frac{1}{2^n}$ for all $n \geq 0$.

(iv) Deduce from (iii) that the sequence $\{C_n\}$ is Cauchy and therefore converges.

Recall that the algorithm for solving Pell's equation using continued fractions was discussed at the end of class on April 22nd. This algorithm is also described in Problems 7-9 here:

`http://www.math.ubc.ca/~gor/pell.pdf`

4. Find a non-trivial solution to Pell's equation $x^2 - dy^2 = 1$ in each of the following cases:

(i) $d = (a^2 - 1)$ for some $a \in \mathbb{N}$

(ii) $d = a^2 + 1$ for some $a \in \mathbb{N}$

(iii) $d = a(a + 1)$ for some $a \in \mathbb{N}$

Hint: In case (i) a solution is easy to guess; in cases (ii) and (iii) one can also guess; alternatively you can use continued fractions.

5. Find a solution Pell's equation $x^2 - 14y^2 = 1$ in two ways:

(i) Computing fractional parts of $k\sqrt{14}$ for small values of k (you may use a calculator)

(ii) Using continued fractions

6. Use continued fractions to find a solution to Pell's equation $x^2 - dy^2 = 1$ for $d = 19$ and $d = 41$.

7. Prove that for every $n \in \mathbb{N}$ there exists a solution to the equation $x^2 - 3y^2 = 1$ satisfying $10^n < x < 10^{n+1}$. **Hint:** How are all solutions obtained from the fundamental solution?

8. Let (x, y, z) be a primitive integer solution for the equation $x^2 + 2y^2 = z^2$. Prove that there exist integers u and v such that $(x, y, z) = (2u^2 - v^2, 2uv, 2u^2 + v^2)$ or $(u^2 - 2v^2, 2uv, u^2 + 2v^2)$.