Homework #1. Due Wednesday, January 23rd, in class

Reading:

1. For this homework assignment: Chapter 1 and Section 2.1. Note that some material in Chapter 1 (like the general solution to a linear diophantine equation) was not explicitly discussed in class, but it is based on ideas similar to the ones we discussed.

2. Before the class on Wed, Jan 23: Section 2.2-2.4.

Problems:

Problem 1: Prove that $6 \mid (n^3 - n)$ for any integer n.

Problem 2: The Fibonacci numbers $f_1, f_2, \ldots$ are defined recursively by $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Prove that f_n and f_{n+1} are coprime for all n.

Problem 3:

(a) Let $d = gcd(123, 321)$. Find d and also find integers u and v such that $d = 123u + 321v$.

(b) Now find all integer pairs (x, y) such that $123x + 321y = 12$.

Problem 4: Let a and b be positive integers and $d = gcd(a, b)$.

(a) Prove that for any integer c such that $c > ab - a - b$ and $d \mid c$ there exist nonnegative integers x and y such that $c = ax + by$. **Hint:** Let x be the smallest nonnegative integer such that $c = ax + by$ for some $y \in \mathbb{Z}$ (explain why such x exists). Show that $x < b$ and deduce that y corresponding to this x is nonnegative.

(b) Assume that a and b are coprime, that is $d = 1$. Prove that $c = ab-a-b$ cannot be written as $c = ax+by$ where x and y are nonnegative integers.

(c) Now assume that a and b are NOT coprime. Prove that $c = ab - a - b$ can be written as $c = ax + by$ for nonnegative integers x and y.

Problem 5: Let $n, m \in \mathbb{Z}$ and suppose that $gcd(n, m) = 1$. Prove that $gcd(n - m, n + m) = 1$ or 2 and show by examples that both possibilities may occur. **Hint:** Use Corollary 1.11(b) to prove that $gcd(n-m, n+m)$ divides 2.

Problem 6: Let $a, b \geq 2$ be integers. Let $p_1, \ldots, p_k$ be the distinct primes which divide a or b and write $a = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ and $b = p_1^{\beta_1} \ldots p_k^{\beta_k}$. Prove that $lcm(a, b) = p_1^{\gamma_1} \ldots p_k^{\gamma_k}$ where $\gamma_i = \max\{\alpha_i, \beta_i\}$.

Problem 7: Let n and m be positive integers. Suppose that nm is a complete square, that is, $nm = a^2$ for some $a \in \mathbb{Z}$.

(a) Assume in addition that n and m are coprime. Prove that both n and m are complete squares.

(b) Show by example that the conclusion of (a) may be false if n and m are not coprime.