Number Theory, Fall 2016. Test #4.
Due on Friday, December 2nd, by 1pm

Directions: Provide complete arguments (do not skip steps). State clearly and FULLY any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given. If you are unable to solve a problem (or a part of a problem), you may still use its result to solve a later part of the same problem or a later problem in the exam.

Rules: You may freely use your class notes, the class textbook by Jones and Jones, all materials from Math 5653 websites (Spring 2013, Spring 2014, Fall 2016), all publicly available materials from Math 3354 website from Spring 2016 as well as the following notes on continued fractions and Pell's equation:
`http://math.stanford.edu/~jbooher/expos/continued_fractions.pdf`

You may also ask me any questions about the problems. You are NOT allowed to

(i) discuss midterm problems with anyone else except me

(ii) use any online resources except the ones mentioned above

(iii) use other books without obtaining a prior permission

Violation of any of the rules (i)-(iii) will be considered a violation of UVA honor code and appropriate action will be taken.

1. In all parts of this problem make sure to include all the calculations.

(a) (4 pts) Find a non-trivial solution to the equation $x^2 - 23y^2 = 1$

(b) (4 pts) Find a non-trivial solution to the equation $x^2 - 53y^2 = 1$

(c) (2 pts) Let $k \in \mathbb{N}$. Compute the continued fraction $[k; k, k, k, \ldots]$

2. (10 pts) In all parts of this problem by a solution we mean an integer solution

(a) Let $d, c \in \mathbb{Z}$ where $d > 0$ and d is not a perfect square. Prove that if the equation $x^2 - dy^2 = c$ has a solution, then it has infinitely many solutions.

(b) Let p be a prime such that $p \equiv 3 \mod 4$. Prove that the equation $x^2 - py^2 = p$ has no solutions.

(c) Assume that $d \in \mathbb{N}$ is not a perfect square and that the continued fraction for $\sqrt{d}$ has **odd** period. Prove that $x^2 - dy^2 = d$ has a solution.

See a hint at the end of test.

3. (10 pts) Find all primitive integer solutions to the equation $x^2 + 3y^2 = z^2$ (as usual (x, y, z) is primitive if $gcd(x, y, z) = 1$). Your answer should have the same general form as the answer to Problem 6 in HW#10 from Spring 2014.

4. (10 pts) Let Λ be the set of all completely multiplicative functions from $\mathbb{N}$ to $\mathbb{C}$, and let Δ be the set of all multiplicative functions $f : \mathbb{N} \to \mathbb{C}$ with the property that $f(n) = 0$ whenever n is not square-free. Recall that according to our definition, a multiplicative (or completely multiplicative) function g must satisfy $g(1) = 1$

(a) Let $h \in \Lambda$, and let $H = h^{-1}$, the Dirichlet inverse of h. Prove that $H(n) = h(n)\mu(n)$ for all n and deduce that $H \in \Delta$ (here $h(n)\mu(n)$ is the regular multiplication).

(b) Now prove that for any $f \in \Delta$, its Dirichlet inverse lies in Λ.

(c) Recall that the set M of all multiplicative functions forms a group with respect to the Dirichlet product. Note that parts (a) and (b) simply say that $\Lambda = \Delta^{-1}$, that is, Λ is precisely the set of inverses of elements of Δ (and vice versa). Now let $\langle \Delta \rangle_+$ be the set of elements of M representable as $f_1 * \ldots * f_k$ with each $f_i \in \Delta$ and $k \geq 1$ (in group-theoretic terminology, $\langle \Delta \rangle_+$ is the semigroup generated by Δ). Prove that the intersection $\langle \Delta \rangle_+ \cap \Lambda$ contains just 1 element, the function I. **Hint:** What can you say about the values of elements of $\langle \Delta \rangle_+$ and Λ on prime powers?

Hint for 2: For (a) and (c) use one of the results about solutions to Pell's equation (you will need different results for (a) and (c)).