Homework #9. Due Thursday, November 17th

Reading:

1. For this homework assignment: Chapter 8 + class notes (Lectures 21, 22)
2. For the next week's classes: Chapter 9

Problems:

1. Let $\mathbb{PP}$ be the set of all prime powers (where 1 is not considered a prime power), and let $f : \mathbb{PP} \to \mathbb{C}$ be an arbitrary function. Extend f to a function $F : \mathbb{N} \to \mathbb{C}$ by setting $f(1) = 1$ and

$$F(p_1^{\alpha_1} \dots p_k^{\alpha_k}) = f({p_1}^{\alpha_1}) \dots f({p_k}^{\alpha_k})$$

whenever $p_1, \dots, p_k$ are distinct primes (so in particular, F restricted to $\mathbb{PP}$ is equal to f). Prove that F is multiplicative.

2. A function $f : \mathbb{N} \to \mathbb{C}$ is called *completely multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$. (Note that the book does not require $f(1) = 1$ in the definition of a multiplicative or completely multiplicative function. However, it is easy to see that the only function which is multiplicative according to the book but does not satisfy $f(1) = 1$ is the zero function).

(a) Formulate and prove the analogue of problem 1 for completely multiplicative functions.

(b) Show by example that the set of completely multiplicative functions is NOT closed under the Dirichlet product $*$

(c) (stronger version of (b)). Let f be a completely multiplicative function different from I (where, as before, $I(1) = 1$ and $I(n) = 0$ for $n > 1$). Prove that $f * f$ is NOT completely multiplicative.

3. Corollary 8.7 from the book can be restated as equality $\phi = \mu * N$ where N is the function defined by $N(n) = n$. In the book Corollary 8.7 is obtained as a consequence of the Euler's identity $\sum_{d|n} \phi(d) = n$. Give a direct proof of Corollary 8.7 using the fact that μ, N and ϕ are all multiplicative and the fact that a multiplicative function is completely determined by its values on prime powers.

4. Exercise 8.21 from the book.

5. Exercise 8.22 from the book.

6. Fix an integer $m \geq 1$. As in class, for $n \geq 1$, we denote by W_n the set of all words of length n in the alphabet with m symbols and by A_n the set of all aperiodic words of length n in the same alphabet. Recall that in class we used Möbius inversion to show that

$$|A_n| = \sum_{d|n} m^d \mu(\tfrac{n}{d}) = \sum_{d|n} \mu(d) m^{\frac{n}{d}}. \qquad (***)$$

(a) Read about the inclusion-exclusion principle (see Exercise 5.10 in the book or look it up online).

(b) Use the inclusion-exclusion principle to prove the formula (***) above. **Hint:** Given $n \in \mathbb{N}$, let $p_1, \ldots, p_k$ be the distinct prime divisors of n, and apply the inclusion-exclusion principle to the sets $W_{n,p_1}, \ldots, W_{n,p_k}$ where $W_{n,l}$ is the set of all words in W_n representable as v^l for some word v. Note that $\cup_{i=1}^k W_{n,p_i}$ is the set of all periodic words of length n (explain why).