Homework #7. Due Thursday, October 27th

Reading:

1. For this homework assignment: Chapter 10, Sections 10.1-10.3 + class notes (Lectures 14-16)

2. For the next week's classes: TBA

Problems:

1. Recall that for a prime p and a nonzero integer n, by $ord_p(n)$ we denote the largest power of p which divides n. Assume now that p is a prime of the form $4k+3$

(a) Prove that if $p \nmid a$ or $p \nmid b$, then $p \nmid (a^2+b^2)$. **Hint:** Assume that $a^2+b^2=pk$, rewrite this equation in a suitable way and then use Legendre symbols to get a contradiction.

(b) Use (a) to prove that $ord_p(a^2+b^2)$ is even for any $a, b \in \mathbb{Z}$ with $a \neq 0$ or $b \neq 0$. This completes the proof of theorem characterizing which integers are representable as sums of two squares.

2.

(a) Determine which primes are representable in the form a^2+2b^2 with $a, b \in \mathbb{Z}$. **Hint:** first test all primes up to, say, 50, to make a conjecture. To prove the conjecture use the fact that $\mathbb{Z}[i\sqrt{2}]$ for the positive direction (primes which can be represented) – this is very similar to what we did in class – and then a suitable analogue of Problem 1 for the negative direction (primes which cannot be represented).

(b) Now describe all integers representable as a^2+2b^2 with $a, b \in \mathbb{Z}$.

3. Prove that $\mathbb{Z}[\sqrt{3}]$ and $\mathbb{Z}[\frac{\sqrt{5}+1}{2}]$ are Euclidean domains. **Hint:** use the norm functions described in Problem 7 of HW#6. In the second case use the restriction of the norm function coming from $\mathbb{Q}[\sqrt{5}]$.

4. Let $R=\mathbb{Z}[\sqrt{5}]$. Prove that 2 considered as an element of R is irreducible but not prime. Deduce that $\mathbb{Z}[\sqrt{5}]$ is not a Euclidean domain. **Hint:** To prove that 2 is not prime find two non-equivalent factorizations of 4 in R. To prove that 2 is irreducible argue by contradiction and use the norm function from Problem 7 of HW#6.