Homework #10. Due Tuesday, December 6th

Reading:

1. For this homework assignment and next week's class: class notes (Lectures 26, 27)

Problems:

1. Let $n \in \mathbb{N}$, and define $f : \mathbb{N} \to \mathbb{C}$ by

$$f(a) = \begin{cases} 1 & \text{if } a \equiv 1 \mod n \\ 0 & \text{otherwise} \end{cases}$$

Note that since f is bounded, the Dirichlet series $L_f(s)$ converges for all $s > 1$. Prove that

$$\lim_{s \to 1^+} (s-1) L_f(s) = \frac{1}{n}.$$

Hint: Use upper and lower Riemann sums to show that

$$\int_1^\infty \frac{1}{x^s}\, dx \le n L_f(s) \le n + \int_1^\infty \frac{1}{x^s}\, dx$$

2. Compute all Dirichlet characters of period n for $n = 5, 7$ and 16. Present the answer in the form of a table. (In Lecture 27 we solved the analogous problem for $n = 8$).

3. Fix $n \in \mathbb{N}$, and define $\chi_0 : \mathbb{N} \to \mathbb{C}$ by

$$\chi_0(a) = \begin{cases} 1 & \text{if } gcd(a, n) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then χ_0 is a Dirichlet character of period n called the *principal character*. The corresponding group character $\phi_0 \in \widehat{U_n}$ is the trivial character (given by $\phi_0(g) = 1$ for all $g \in U_n$).

In Lecture 28 we will show that Dirichlet's theorem on primes in arithmetic progressions follows easily from the facts established in previous lectures and the following result:

Let χ be a non-principal Dirichlet character of period n (that is, $\chi \neq \chi_0$). Then

(1) $L_\chi(s)$ is defined and differentiable for all $s > 0$

(2) $L_\chi(1) \neq 0$

The proof of part (2) for general n and χ is the most difficult part of Dirichlet's theorem.

Use your answer in Problem 2 to prove that (2) holds for every $\chi \neq \chi_0$ for $n = 5$ and $n = 7$. **Hint:** In those cases it is actually true that $\mathrm{Re} L_\chi(1) > 0$.