Math 5651. Fall 2011. Solutions to the in-class part of the second midterm.

1. Let V be a finite-dimensional vector space and $n = \dim(V)$. Let $S, T \in \mathcal{L}(V)$ be s.t.

$$ST = 0$$

Prove that

$$\mathrm{rk}(S) + \mathrm{rk}(T) \leq n$$

referring only to results proved in class.

Solution: Since $ST = 0$, we have $S(T(v)) = 0$ for all $v \in V$, and thus, $\mathrm{Im}\,(T) = \{T(v) : v \in V\} \subseteq \mathrm{Ker}\,(S)$. Therefore, $\mathrm{rk}(T) = \dim(\mathrm{Im}\,(T)) \leq \dim(\mathrm{Ker}\,(S)) = \mathrm{null}(S)$, and using the rank-nullity theorem we get

$$\mathrm{rk}(S) + \mathrm{rk}(T) \leq \mathrm{rk}(S) + \mathrm{null}(S) = \dim(V) = n.$$

2. Let A be a 3×3 matrix over $\mathbb{R}$. Eight of the nine entries of A are given below:

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & ? \\ 3 & 9 & -3 \end{pmatrix}$$

Let $\alpha \neq 0$ be a fixed real number, and suppose that one of the eigenvalues of A is equal to α.

(a) Find all the other eigenvalues of A. **Hint:** This can be done almost without computations.

(b) List all values of α for which A is not diagonalizable. Make sure to prove that for each α you listed A is not diagonalizable and for each α you did not list A is diagonalizable.

Solution: (a) First note that the first two columns of A are proportional, so $\det(A) = 0$. Therefore, 0 is an eigenvalue of A (e.g. since $\det(A) = \det(A - 0 \cdot I) = \chi_A(0)$). We also know that the sum of all eigenvalues of A is equal to $\mathrm{tr}(A) = 1 + 6 - 3 = 4$. Thus, the eigenvalues of A are $\alpha, 0$ and $4 - \alpha - 0 = 4 - \alpha$.

(b) If A has 3 distinct eigenvalues, it is diagonalizable by Corollary 16.5. Therefore, A may fail to be diagonalizable only if the numbers $0, \alpha$ and $4 - \alpha$

are not distinct, which happens if $0 = 4 - \alpha$ (in which case $\alpha = 4$) or if $\alpha = 4 - \alpha$ (in which case $\alpha = 2$).

If $\alpha = 4$, we have $m_4(A) = 1$ and $m_0(A) = 2$, so A is diagonalizable $\iff \dim E_0(A) = \text{null}(A) = 2$ which is equivalent to saying that $\text{rk}(A) = 3 - \text{null}(A) = 1$. But clearly $\text{rk}(A) \geq 2$ since the first and third rows of A are not proportional. Therefore, A is not diagonalizable if $\alpha = 0$.

Similarly, if $\alpha = 2$, we have $m_2(A) = 2$ but $\text{rk}(A - 2I) \geq 2$ (for the same reason), so A is not diagonalizable if $\alpha = 2$ as well.

Final answer for (b): $\alpha = 2, 4$.

3. Let A be an $n \times n$ matrix over a field F. For each $1 \leq k, l \leq n$ by a $k \times l$ **submatrix of** A we mean the object obtained from A by removing $n - k$ rows and $n - l$ columns (there are no restrictions on which rows and columns are being removed).

(a) Suppose that A has a $k \times k$ submatrix B with $\det(B) \neq 0$. Prove that $rk(A) \geq k$.

(b) Conversely, suppose that $rk(A) \geq k$. Prove that there exists a $k \times k$ submatrix B with $\det(B) \neq 0$. **Hint:** What can you say about possible ranks of $n \times k$ submatrices of A?

Solution: (a) WOLOG, the matrix B lies at the intersection of the first k rows and the first k columns of A. Since $\det(B) \neq 0$, the columns of B are linearly independent. This implies that the first k columns of A are linearly independent. Indeed, suppose that there exist $\lambda_1, \ldots, \lambda_k \in F$, not all 0 s.t. $\lambda_1 col_1(A) + \ldots + \lambda_k col_k(A) = 0$. This means that $\lambda_1 a_{i1} + \ldots + \lambda_k a_{ik} = 0$ for all $1 \leq i \leq m$. Forgetting about $i \geq k + 1$ and using these equalities just for $1 \leq i \leq k$, we conclude that $\lambda_1 col_1(B) + \ldots + \lambda_k col_k(B) = 0$, contrary to our assumption.

Thus, A has k linearly independent columns, so $\text{rk}(A) \geq k$.

(b) Since $\text{rk}(A) \geq k$, A has k linearly independent columns. Consider the $m \times k$ matrix C composed of those k columns. Then C also has k linearly independent columns, so $\text{rk}(C) = k$ (it cannot be larger than k since C has the total of k columns). Since the rank of a matrix is also equal to the largest number of linearly independent rows, we conclude that C has k linearly independent rows. Let B be the matrix composed of those k rows of C. By construction, B is a $k \times k$ submatrix of A and B has k linearly independent rows, so $\text{rk}(B) = k$ and hence $\det(B) \neq 0$.

4. Let V be a six-dimensional vector space over a field F. Let $T \in \mathcal{L}(V)$ and assume that $\chi_T(x) = (x - \lambda)^4 (x - \mu)^2$ for some $\lambda \neq \mu$.

(a) List all possibilities for $JCF(T)$ (up to permutation of blocks). An answer is sufficient. To save time instead of writing down matrices just write which Jordan blocks they are composed of, denoting by $J(\alpha, k)$ the Jordan block of size k corresponding to α.

(b) Now assume in addition that $\dim E_\lambda(T) = 2$ and $\dim E_\mu(T) = 1$. List all possibilities for $JCF(T)$. For each JCF that you listed in (a) but not in (b) explain why it cannot occur in (b).

(c) Keeping all the previous assumptions, suppose in addition that V CANNOT be written as $V = U \oplus W$ where U and W are both T-invariant and $\dim(U) = \dim(W) = 3$. Find $JCF(T)$ (it is uniquely determined by the given information).

Solution: (a) There are 2 ways to write 2 as sum of positive integers (where the order does not matter): $2 = 2$ and $2 = 1 + 1$ and 5 ways to write 4 in such a way: $4 = 4$, $4 = 3 + 1$, $4 = 2 + 2$, $4 = 2 + 1 + 1$, $4 = 1 + 1 + 1 + 1$. Therefore, there are $10 = 5 \cdot 2$ possibilities for $JCF(T)$;

(i) $JCF(T) = J(\lambda, 4) \oplus J(\mu, 2)$

(ii) $JCF(T) = J(\lambda, 3) \oplus J(\lambda, 1) \oplus J(\mu, 2)$

(iii) $JCF(T) = 2J(\lambda, 2) \oplus J(\mu, 2)$

(iv) $JCF(T) = J(\lambda, 2) \oplus 2J(\lambda, 1) \oplus J(\mu, 2)$

(v) $JCF(T) = 4J(\lambda, 1) \oplus J(\mu, 2)$

(vi) $JCF(T) = J(\lambda, 4) \oplus 2J(\mu, 1)$

(vii) $JCF(T) = J(\lambda, 3) \oplus J(\lambda, 1) \oplus 2J(\mu, 1)$

(viii) $JCF(T) = 2J(\lambda, 2) \oplus 2J(\mu, 1)$

(ix) $JCF(T) = J(\lambda, 2) \oplus 2J(\lambda, 1) \oplus 2J(\mu, 1)$

(x) $JCF(T) = 4J(\lambda, 4) \oplus 2J(\mu, 1)$

(b) We start with a general claim:

Claim: *For each $\alpha \in Spec(T)$, the dimension of the eigenspace $E_\alpha(T)$ is equal to the number of Jordan blocks corresponding to α.*

Proof: This claim follows from our proof of existence of JCF, but can be also seen directly as follows. We have $\dim E_\alpha(T) = \text{null}(T - \alpha I) = 6 - \text{rk}(T - \alpha I) = 6 - \text{rk}(A - \alpha I_6)$ where $A = JCF(T)$. If c_α is the number of blocks in A corresponding to α, then the matrix $A - \alpha I_6$ has c_α columns which are identically zero (the first column of each Jordan block). The remaining columns are nonzero and moreover have different degrees, if we define the degree of a column to be the largest i s.t. the column has nonzero entry in the i^{th} row. The latter property easily implies that the $6 - c_\alpha$ nonzero columns of $A - \alpha I_6$ are linearly independent. So, $\text{rk}(A - \alpha I_6) = 6 - c_\alpha$, whence $\dim E_\alpha(T) = 6 - (6 - c_\alpha) = c_\alpha$, as claimed above. $\square$

According to the claim, conditions $\dim E_\lambda(T) = 2$ and $\dim E_\mu(T) = 1$ hold $\iff$ $JCF(T)$ has two blocks corresponding to λ and one block corresponding to μ. Therefore, of the 10 possibilities from (a) precisely two, namely (ii) and (iii), may occur in (b).

(c) If $JCF(T) = J(\lambda, 3) \oplus J(\lambda, 1) \oplus J(\mu, 2)$ (option (ii)) and β is a corresponding Jordan basis, we can write $V = U \oplus W$ where U is the span of the first three vectors of β (corresponding to the block $J(\lambda, 3)$) and W is the span of the last three vectors of β (corresponding to the other two blocks). Then $\dim(U) = \dim(W) = 3$ and both U and W are T-invariant. Thus, option (ii) cannot occur, and the only possibility is (iii): $JCF(T) = 2J(\lambda, 2) \oplus J(\mu, 2)$.

Note: It takes a little work to show that option (iii) does satisfy the requirement in (c), but this was not a part of the problem.