NAME:

Math 5651. Advanced Linear Algebra. Fall 2011. Second midterm (in-class part)
Tuesday, November 8th, 2-3:20 pm

Directions: No books, notes, calculators, laptops, PDAs, cellphones, web appliances, or similar aids are allowed. All work must be your individual efforts.

- Show all your work and justify all statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

Scoring system: Exam consists of **4** problems, each of which is worth **7** points. Your regular total is the sum of the best **3** out of **4** scores (so the maximum regular total is 21). If k is the lowest of your 4 scores and $k > 4$, you will get $k-4$ bonus points (so the maximum total with the bonus is 24).

problem	1	2	3	4	total	B	sum
points	7	7	7	7	21	3	
score							

1. Let V be a finite-dimensional vector space and $n = \dim(V)$. Let $S, T \in \mathcal{L}(V)$ be s.t.

$$ST = 0$$

Prove that

$$rk(S) + rk(T) \leq n$$

referring only to results proved in class.

Hint: Use the rank-nullity theorem.

2. Let A be a 3×3 matrix over $\mathbb{R}$. Eight of the nine entries of A are given below:

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & ? \\ 3 & 9 & -3 \end{pmatrix}$$

Let $\alpha \neq 0$ be a fixed real number, and suppose that one of the eigenvalues of A is equal to α.

(a) Find all the other eigenvalues of A. **Hint:** This can be done almost without computations.

(b) List all values of α for which A is not diagonalizable. Make sure to prove that for each α you listed A is not diagonalizable and for each α you did not list A is diagonalizable.

3. Let A be an $n \times n$ matrix over a field F. For each $1 \leq k, l \leq n$ by a $k \times l$ **submatrix of** A we mean the object obtained from A by removing $n - k$ rows and $n - l$ columns (there are no restrictions on which rows and columns are being removed).

(a) Suppose that A has a $k \times k$ submatrix B with $\det(B) \neq 0$. Prove that $rk(A) \geq k$.

(b) Conversely, suppose that $rk(A) \geq k$. Prove that there exists a $k \times k$ submatrix B with $\det(B) \neq 0$. **Hint:** What can you say about possible ranks of $n \times k$ submatrices of A?

4. Let V be a six-dimensional vector space over a field F. Let $T \in \mathcal{L}(V)$ and assume that $\chi_T(x) = (x - \lambda)^4 (x - \mu)^2$ for some $\lambda \neq \mu$.

(a) List all possibilities for $JCF(T)$ (up to permutation of blocks). An answer is sufficient. To save time instead of writing down matrices just write which Jordan blocks they are composed of, denoting by $J(\alpha, k)$ the Jordan block of size k corresponding to α.

(b) Now assume in addition that $\dim E_\lambda(T) = 2$ and $\dim E_\mu(T) = 1$. List all possibilities for $JCF(T)$. For each JCF that you listed in (a) but not in (b) explain why it cannot occur in (b).

(c) Keeping all the previous assumptions, suppose in addition that V CANNOT be written as $V = U \oplus W$ where U and W are both T-invariant and $\dim(U) = \dim(W) = 3$. Find $JCF(T)$ (it is uniquely determined by the given information).

Extra page.