Advanced Linear Algebra, Fall 2011. Midterm #2, take-home part.

DUE THURSDAY, NOVEMBER 10TH

Directions: Provide complete arguments (do not skip steps). State clearly and FULLY any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given. If you are unable to solve a problem (or a part of a problem), you may still use its result to solve a later part of the same problem or a later problem in the exam.

Scoring system: Exam consists of **3** problems, all of which will be counted. The maximal regular total is 32 points. In addition, you may get up to 4 bonus points by solving the last part of Problem 2.

Rules: You are NOT allowed to discuss midterm problems with anyone else except me. You may ask me any questions about the problems (e.g. if the formulation is unclear), and I may provide some very minor hints. You may freely use your class notes, previous homework assignments, and the class textbook by Friedberg, Insel and Spence. The use of other books or any online sources is not allowed.

1. Let F be a field, $A \in Mat_{m\times n}(F)$ for some m and n, and let $k = rk(A)$.

(a) (2 pts) Prove that if $A = A_1 + \ldots + A_l$ where $rk(A_i) = 1$ for each i, then $l \geq k$.

(b) (4 pts) Prove that there exist matrices $A_1, \ldots, A_k$, with $rk(A_i) = 1$ for each i, such that $A = A_1 + \ldots + A_k$.

2. Let V be a finite-dimensional vector space over a field F and let $T \in \mathcal{L}(V)$ be such that $\chi_T(x)$ splits. Let $n = \dim(V)$.

(a) (5 pts) Suppose that T has n distinct eigenvalues. Prove that V has precisely 2^n T-invariant subspaces.

(b) (5 pts) Suppose that there exists an ordered basis $\beta = \{v_1, \ldots, v_n\}$ of V s.t. $[T]_\beta$ is a Jordan block of size n corresponding to $\lambda = 0$ (equivalently $(v_1, \ldots, v_n)$ is a nilpotent T-cycle). Prove that V has precisely $n + 1$ T-invariant subspaces. **Hint:** A T-invariant subspace W is called T-cyclic if there exists a vector $v \in W$ s.t. W is the smallest T-invariant subspace containing v (see § 5.4). First describe all possible T-cyclic subspaces and then prove that every T-invariant subspace is T-cyclic (under the given assumptions on T).

(c) (2 pt) Give an example (with proof) where V has infinitely many T-invariant subspaces and T is not scalar, that is, $T \neq \lambda I$ for any $\lambda \in F$.

(d) (2 pts) Give an example (without proof) where $n = 3$ and V has precisely **six** T-invariant subspaces. **Hint:** Use (a) and (b).

(e) (4 pts, **bonus**) Give a proof in your example from (d).

3. Let F be a field, $A \in Mat_{n\times n}(F)$, and assume that $\chi_A(x)$ splits. Let $\lambda_1, \ldots, \lambda_t$ be the distinct eigenvalues of A, and let m_i be the multiplicity of λ_i.

(a) (4 pts) Prove that $tr(A^k) = m_1\lambda_1^k + \ldots + m_t\lambda_t^k$ for each $k \in \mathbb{Z}_{>0}$.

(b) (6 pts) Assume that $F = \mathbb{R}$ (real numbers) and $tr(A^k) = 0$ for all $k \in \mathbb{Z}_{>0}$. Prove that $t = 1$ and $\lambda_1 = 0$ (that is, A has just one eigenvalue and that eigenvalue is 0). Deduce that A is nilpotent.

(c) (2 pts) Does the assertion of (b) remain true if $\mathbb{R}$ is replaced by an arbitrary field F? Prove or give a counterexample.