Advanced Linear Algebra, Fall 2011. Solutions to midterm #1.

1. Let $V = P_2(\mathbb{R})$, the vector space of polynomials of degree ≤ 2 over $\mathbb{R}$. Let $T : V \to V$ be the differentiation map, that is, $T(f(x)) = f'(x)$.

(a) Find the matrix $[T]_\beta$ with respect to the ordered basis $\beta = \{x^2, 2x, x^2 + 2x + 2\}$ of V. (You need not prove that T is linear or that β is a basis).

(b) Prove that there is NO ordered basis γ of V s.t.

$$[T]_\gamma = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{pmatrix}.$$

Solution: (a) We have

$$\begin{aligned} &T(x^2) = 2x = 0 \cdot x^2 + 1 \cdot 2x + 0 \cdot (x^2 + 2x + 2), \\ &T(2x) = 2 = (-1) \cdot x^2 + (-1) \cdot 2x + 1 \cdot (x^2 + 2x + 2), \\ &T(x^2 + 2x + 2) = 2x + 2 = (-1) \cdot x^2 + 0 \cdot 2x + 1 \cdot (x^2 + 2x + 2). \end{aligned}$$

Therefore,

$$[T]_\gamma = \begin{pmatrix} 0 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

(b) Suppose there exists a basis $\gamma = \{f(x), g(x), h(x)\}$ s.t.

$$[T]_\gamma = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{pmatrix}.$$

Then $T(f(x)) = 1 \cdot f(x) + 0 \cdot g(x) + 0 \cdot h(x) = f(x)$, so $f'(x) = f(x)$. If $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$, so $\deg(f'(x)) < \deg(f(x))$ unless $f(x) = 0$. Thus we conclude that $f(x) = 0$, which is impossible since 0 cannot be contained in any linearly independent set (in particular, any basis).

2. Let $f : A \to B$ be a function from a set A to a set B. As usual, for a subset C of A, we let $f(C) = \{f(c) : c \in C\}$ be the image of C inder f. If

D is a subset of B, the **preimage of D under f** is the subset $f^{-1}(D) \subseteq A$ defined by

$$f^{-1}(D) = \{a \in A : f(a) \in D\}.$$

Note that f^{-1} here is NOT the inverse function (which may not exist) – $f^{-1}(D)$ should be treated as a single expression.

Now let V and W be vector spaces over the same field, let $T : V \to W$ a linear map, and let U be a subspace of W.

(a) Prove that $T^{-1}(U)$ is a subspace of V.

(b) Prove that $T(T^{-1}(U)) \subseteq U$. **Hint:** not much to do here.

(c) Give an example showing that $T(T^{-1}(U))$ may not equal U.

(d) Now assume that V and W are finite-dimensional. Prove that $\dim(T^{-1}(U)) \leq \dim(U) + \dim(\mathrm{Ker}(T))$.

Solution: (a) As usual, we check three conditions:

Since $T(0) = 0 \in U$, we have $0 \in T^{-1}(U)$, so $T^{-1}(U)$ contains 0.

If $v, w \in T^{-1}(U)$, then $T(v), T(w) \in U$, and since U is a subspace, $T(v)+T(w) \in U$. But $T(v)+T(w) = T(v+w)$, so $T(v+w) \in U$, and thus $v+w \in T^{-1}(U)$. Hence $T^{-1}(U)$ is closed under addition.

Finally, if $v \in T^{-1}(U)$ and $\lambda \in F$, then $T(v) \in U$, so $T(\lambda v) = \lambda T(v) \in U$ (as U is a subspace). Hence $\lambda v \in T^{-1}(U)$, so U is closed under scalar multiplication.

(b) We have $T(T^{-1}(U)) = \{T(v) : v \in T^{-1}(U)\} = \{T(v) : T(v) \in U\} \subseteq U$.

(c) Let $U = W$ and $T : V \to W$ any non-surjective linear map. Then $T(T^{-1}(U)) \subseteq T(V) \neq U$. For instance, we can let F to be any field, $U = V = W = F$ and let $T : V \to W$ be the trivial map, that is, $T(v) = 0$ for all $v \in V$.

(d) Consider the map $T' : T^{-1}(U) \to U$ given by $T'(v) = T(v)$. Note that $\mathrm{Ker}(T') = \mathrm{Ker}(T)$. Indeed, since T' is obtained from T by restricting the domain to $T^{-1}(U)$, we have $\mathrm{Ker}(T') = \mathrm{Ker}(T) \cap T^{-1}(U)$. Since $T^{-1}(U) \supseteq T^{-1}(\{0\}) = \mathrm{Ker}(T)$, so $\mathrm{Ker}(T) \cap T^{-1}(U) = \mathrm{Ker}(T)$.

Applying the rank-nullity theorem to T', we get

$$\dim(T^{-1}(U)) = \dim(\mathrm{Ker}(T')) + \dim(\mathrm{Im}(T')) \leq \dim(\mathrm{Ker}(T)) + \dim(U)$$

since $\mathrm{Ker}(T') = \mathrm{Ker}(T)$ and $\mathrm{Im}(T') \subseteq U$ by construction.

3. Let $V = P_4(\mathbb{R})$.

(a) Prove that the set $\{x^4, x^4 + x^3, x^4 + x^2, x^4 + x, x^4 + 1\}$ is a basis of V.

(b) Prove that the set

$$\{(x-1)(x-2)(x-3)(x-4), (x-1)(x-2)(x-3)(x-5), (x-1)(x-2)(x-4)(x-5), (x-1)(x-3)(x-4)(x-5), (x-2)(x-3)(x-4)(x-5)\}$$

is a basis of V.

Solution: (a) Since the given set contains 5 elements and $5 = \dim(V)$, it is enough to prove that the elements are linearly independent.

So suppose that $ax^4 + b(x^4 + x^3) + c(x^4 + x^2) + d(x^4 + x) + e(x^4 + 1) = 0$ for some $a, b, c, d, e \in \mathbb{R}$. Putting the left-hand side into standard form, we get $(a + b + c + d + e)x^4 + bx^3 + cx^2 + dx + e = 0$. By definition two polynomials are equal $\iff$ they have the same coefficient of x^n for each n. Thus, we conclude that $a + b + c + d + e = 0$ and $b = c = d = e = 0$. Substituting $b = c = d = e = 0$ in the first equality, we get that $a = 0$ as well. Therefore, the set $\{x^4, x^4 + x^3, x^4 + x^2, x^4 + x, x^4 + 1\}$ is linearly independent.

(b) Denote the given polynomials by $p_1(x), p_2(x), p_3(x), p_4(x), p_5(x)$ in the order they are given in the problem. As in (a), it suffices to check linear independence of $p_1, \ldots, p_5$. So, suppose that for some $\alpha_1, \ldots, \alpha_5 \in \mathbb{R}$ we have

$$\sum_{i=1}^{5} \alpha_i p_i(x) = 0 \qquad (***)$$

The fact that (***) holds as equality of polynomials implies that it also holds as equality of functions, that is, plugging in any real number for x in (***) should give us a valid a numerical equality. Note that $p_2(5) = p_3(5) = p_4(5) = p_5(5) = 0$ while $p_1(5) = (-1)(-2)(-3)(-4) = 24 \neq 0$. Thus, setting $x = 5$ in (***), we get $24\alpha_1 = 0$, so $\alpha_1 = 0$

Similarly, setting $x = 4$, $x = 3$, $x = 2$ and $x = 1$, we obtain that $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 0$ and $\alpha_5 = 0$, respectively. Thus each $\alpha_i = 0$, so $\{p_1, \ldots, p_5\}$ are linearly independent.

4. Let V be a finite-dimensional vector space and U_1, U_2, U_3 subspaces of V.

(a) Prove that $(U_1 \cap U_3) + (U_2 \cap U_3) \subseteq (U_1 + U_2) \cap U_3$

(b) Prove that $\dim(U_1+U_2+U_3) \leq \dim(U_1)+\dim(U_2)+\dim(U_3)-\dim(U_1\cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3)$.

(c) Give an example showing that the inequality in (b) may be strict.

Solution: (a) By definition, any element of $(U_1 \cap U_3) + (U_2 \cap U_3)$ is of the form $x + y$ with $x \in U_1 \cap U_3$ and $y \in U_2 \cap U_3$. Then $x + y \in U_1 + U_2$ (since $x \in U_1$ and $y \in U_2$) and $x + y \in U_3$ (since $x, y \in U_3$ and U_3 is a subspace), so $x + y \in (U_1 + U_2) \cap U_3$.

(b) First note that $U_1 + U_2 + U_3 = (U_1 + U_2) + U_3$. Applying the formula

$$\dim(X + Y) = \dim(X) + \dim(Y) - \dim(X \cap Y),$$

first with $X = U_1 + U_2$ and $Y = U_3$ and then with $X = U_1$ and $Y = U_2$, we get

$$\begin{aligned}\dim(U_1 + U_2 + U_3) &= \dim(U_1 + U_2) + \dim(U_3) - \dim((U_1 + U_2) \cap U_3) = \\ &\dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2) + \dim(U_3) - \dim((U_1 + U_2) \cap U_3)\end{aligned}$$

Since $(U_1 \cap U_3) + (U_2 \cap U_3) \subseteq (U_1 + U_2) \cap U_3$, we have $\dim((U_1 \cap U_3) + (U_2 \cap U_3)) \leq \dim((U_1 + U_2) \cap U_3)$, so $-\dim((U_1 + U_2) \cap U_3) \leq -\dim((U_1 \cap U_3) + (U_2 \cap U_3))$. Combining this with the above formula for $\dim(U_1 + U_2 + U_3)$, we get the desired inequality.

(c) The proof of (b) shows that the inequality in (b) will be strict whenever $(U_1 \cap U_3) + (U_2 \cap U_3) \neq (U_1 + U_2) \cap U_3$.

As a simple example where the latter occurs, we can take $V = F^2$ (where F is any field), $U_1 = \text{Span}(e_1)$, $U_2 = \text{Span}(e_2)$ and $U_3 = \text{Span}(e_1 + e_2)$. Then $U_1 \cap U_3 = U_2 \cap U_3 = \{0\}$, so $(U_1 \cap U_3) + (U_2 \cap U_3) = \{0\}$ while $U_1 + U_2 = F^2$, so $(U_1 + U_2) \cap U_3 = U_3$. We can also check directly that inequality in (b) is strict: $\dim(U_1 + U_2 + U_3) = \dim(F^2) = 2$, while $\dim(U_1) + \dim(U_2) + \dim(U_3) - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3) = 1 + 1 + 1 - 0 - 0 - 0 + 0 = 3$.

5. Given a vector space V and an integer $k \geq 1$, denote by $Sub_k(V)$ the set of all k-dimensional subspaces of V.

(a) Let V and W be isomorphic finite-dimensional vector spaces over the same field. For each $k \geq 0$ construct a bijection between the sets $Sub_k(V)$ and $Sub_k(W)$. Make sure to prove that your map is indeed a bijection and has correct domain and codomain.

(b) Let V be a finite-dimensional vector space, and let $n = \dim(V)$. Prove that for any integer k between 0 and $\dim(V)$ there is a natural **injective** map $\phi : Sub_k(V) \to Sub_{n-k}(V^*)$ where V^* is the dual space.

Solution: (a) Since V and W are isomorphic, there exists an isomorphism $T : V \to W$. Define the map $\Phi : Sub_k(V) \to Sub_k(W)$ given by

$$\Phi(X) = T(X) \qquad (***)$$

Note that on the left-hand side of (***), X is treated as a single element (of the set $Sub_k(V)$) while on the right-hand side X is treated as a subset of V (so by definition $T(X) = \{T(x) : x \in X\}$).

We will show that Φ is a bijective map, but first we need to check that Φ is indeed a map from $Sub_k(V)$ to $Sub_k(W)$, that is, if X is a k-dimensional subspace of V, then $T(X)$ is a k-dimensional subspace of W.

Given a subspace X of V, define $T' : X \to T(X)$ by $T'(x) = T(x)$ for all $x \in X$. Then T' is linear and injective (since T is linear and injective) and T' is surjective (by the choice of codomain). Thus, T' is an isomorphism from X to $T(X)$, so $T(X) \cong X$. Since isomorphic vector spaces have the same dimension, we conclude that $\dim(T(X)) = \dim(X)$, as desired.

To prove bijectivity of Φ, we explicitly construct the inverse map. Define $\Psi : Sub_k(W) \to Sub_k(V)$ by $\Psi(Y) = T^{-1}(Y)$ (where $T^{-1} : W \to V$ is the inverse of T). Since T^{-1} is also an isomorphism, by the same argument as above, Ψ indeed maps $Sub_k(W)$ to $Sub_k(V)$. Since $\Psi(\Phi(X)) = T^{-1}(T(X)) = X$ for any $X \in Sub_k(V)$ and $\Phi(\Psi(Y)) = T(T^{-1}(Y)) = Y$ for any $Y \in Sub_k(W)$, Ψ is indeed the inverse of Φ, so Φ is bijective.

(b) Define the map $\phi : Sub_k(V) \to Sub_{n-k}(V^*)$ by $\phi(X) = Ann(X)$, the annihilator of X. Since $\dim(Ann(X)) = \dim(V) - dim(X) = n - \dim(X)$ by HW#5.6, ϕ is indeed a map from $Sub_k(V)$ to $Sub_{n-k}(V^*)$.

To prove injectivity, we need to show that if $\text{Ann}(X) = \text{Ann}(Y)$, then $X = Y$. This can be done in many different ways using one of the parts of HW#5.5. Here is one possible argument.

Suppose $X, Y \in Sub_k(V)$ are s.t. $\text{Ann}(X) = \text{Ann}(Y)$. Then $\text{Ann}(\text{Ann}(X) = \text{Ann}(\text{Ann}(Y))$, so by HW#5.5(vi), $\text{Span}(\iota(X)) = \text{Span}(\iota(Y))$ where ι is the canonical isomorphism from V to V^{**}. It is easy to see that for any linear map $f : V \to W$ and any subset S of V one has $\text{Span}(f(S)) = f(\text{Span}(S))$. Thus, we conclude that $\iota(\text{Span}(X)) = \iota(\text{Span}(Y))$. Since ι is injective, applying ι^{-1} to both sides, we get $\text{Span}(X) = \text{Span}(Y)$. Finally, since X and Y are subspaces, $\text{Span}(X) = X$ and $\text{Span}(Y) = Y$, and therefore $X = Y$, as desired.

6. Let p be a prime, $\mathbb{Z}_p$ the field of congruence classes $\mod p$ and V a vector space over $\mathbb{Z}_p$ with $\dim(V) = n < \infty$.

(a) Prove that $|Sub_k(V)| = |Sub_{n-k}(V)|$ for all $0 \leq k \leq n$.

(b) Assume that $n = 3$. Find the total number of subspaces of V (of all possible dimensions).

Solution: (a) By Problem 5(b) there is an injective map from $Sub_k(V)$ to $Sub_{n-k}(V^*)$, whence $|Sub_k(V)| \leq |Sub_{n-k}(V^*)|$. Since $V^* \cong V$, by 5(a) we have $|Sub_{n-k}(V^*)| = |Sub_{n-k}(V)|$, so

$$|Sub_k(V)| \leq |Sub_{n-k}(V)|. \qquad (***)$$

Applying (***) with k replaced by $n-k$, we get $|Sub_{n-k}(V)| \leq |Sub_{n-(n-k)}(V)| = |Sub_k(V)|$. Combining this inequality with (***), we get that $|Sub_{n-k}(V)| = |Sub_k(V)|$.

(b) It is clear that $|Sub_0(V)| = |Sub_3(V)| = 1$, and by part (a), $|Sub_2(V)| = |Sub_1(V)|$, so it suffices to compute $|Sub_1(V)|$, the number of one-dimensional subspaces of V.

Every one-dimensional subspace of V is equal to $\text{Span}(v)$ for some nonzero $v \in V$, and conversely for any nonzero $v \in V$, $\text{Span}(v)$ is a one-dimensional subspace. Moreover, $\text{Span}(v) = \text{Span}(w) \iff w$ is a nonzero scalar multiple of v.

As explained in the solution to HW#2.1, there are $p^3 - 1$ nonzero vectors in V, and any nonzero $v \in V$ has precisely $p - 1$ nonzero multiples (namely $v, 2v, \ldots, (p-1)v$). Hence the total number of distinct one-dimensional subspaces is equal to $\frac{p^3-1}{p-1} = p^2+p+1$. Therefore, the total number of subspaces in V is equal to $2 + 2(p^2 + p + 1) = 2p^2 + 2p + 4$.