Advanced Linear Algebra, Fall 2011. Midterm #1. Due Thursday, October 6th

Directions: Provide complete arguments (do not skip steps). State clearly and FULLY any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given. If you are unable to solve a problem (or a part of a problem), you may still use its result to solve a later part of the same problem or a later problem in the exam.

Scoring system: Exam consists of **6** problems, each of which is worth **10** points. Your regular total is the sum of the best **5** out of **6** scores (so the maximum regular total is 50). If k is the lowest of your 6 scores and $k > 6$, you will get $k-6$ bonus points (so the maximum total with the bonus is 54).

Rules: You are NOT allowed to discuss midterm problems with anyone else except me. You may ask me any questions about the problems (e.g. if the formulation is unclear), but as a rule I will not provide hints. You may freely use your class notes, previous homework assignments, and the class textbook by Friedberg, Insel and Spence. The use of other books or any online sources is not allowed.

1. Let $V = P_2(\mathbb{R})$, the vector space of polynomials of degree ≤ 2 over $\mathbb{R}$. Let $T : V \to V$ be the differentiation map, that is, $T(f(x)) = f'(x)$.

(a) (6 pts) Find the matrix $[T]_\beta$ with respect to the ordered basis $\beta = \{x^2, 2x, x^2 + 2x + 2\}$ of V. (You need not prove that T is linear or that β is a basis).

(b) (4 pts) Prove that there is NO ordered basis γ of V s.t.

$$[T]_\gamma = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{pmatrix}.$$

2. Let $f : A \to B$ be a function from a set A to a set B. As usual, for a subset C of A, we let $f(C) = \{f(c) : c \in C\}$ be the image of C inder f. If D is a subset of B, the **preimage of** D **under** f is the subset $f^{-1}(D) \subseteq A$ defined by

$$f^{-1}(D) = \{a \in A : f(a) \in D\}.$$

Note that f^{-1} here is NOT the inverse function (which may not exist) – $f^{-1}(D)$ should be treated as a single expression.

Now let V and W be vector spaces over the same field, let $T : V \to W$ a linear map, and let U be a subspace of W.

(a) (3 pts) Prove that $T^{-1}(U)$ is a subspace of V. Make sure not to skip any steps.

(b) (1 pt) Prove that $T(T^{-1}(U)) \subseteq U$. **Hint:** not much to do here.

(c) (2 pts) Give an example showing that $T(T^{-1}(U))$ may not equal U. **Hint:** If you cannot find an example, most likely you misunderstand some of the definitions.

(d) (4 pts) Now assume that V and W are finite-dimensional. Prove that $\dim(T^{-1}(U)) \leq \dim(U)+\dim(\mathrm{Ker}(T))$. **Hint:** Use (b) and guess which theorem.

3. Let $V = P_4(\mathbb{R})$.

(a) (5 pts) Prove that the set $\{x^4, x^4+x^3, x^4+x^2, x^4+x, x^4+1\}$ is a basis of V.

(b) (5 pts) Prove that the set

$$\{(x-1)(x-2)(x-3)(x-4), (x-1)(x-2)(x-3)(x-5), (x-1)(x-2)(x-4)(x-5), \\ (x-1)(x-3)(x-4)(x-5), (x-2)(x-3)(x-4)(x-5)\}$$

is a basis of V. **Hint:** Do not try to put any of these polynomials into standard form. This will lead to a long, ugly, and unnecessary calculation.

4. Let V be a finite-dimensional vector space and U_1, U_2, U_3 subspaces of V.

(a) (2 pt) Prove that $(U_1 \cap U_3) + (U_2 \cap U_3) \subseteq (U_1 + U_2) \cap U_3$

(b) (5 pts) Prove that $\dim(U_1+U_2+U_3) \leq \dim(U_1)+\dim(U_2)+\dim(U_3) - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3)$.

(c) (3 pts) Give an example showing that the inequality in (b) may be strict.

5. Given a vector space V and an integer $k \geq 1$, denote by $Sub_k(V)$ the set of all k-dimensional subspaces of V.

(a) (4 pts) Let V and W be isomorphic finite-dimensional vector spaces over the same field. For each $k \geq 0$ construct a bijection between the sets $Sub_k(V)$ and $Sub_k(W)$. Make sure to prove that your map is indeed a bijection and has correct domain and codomain. **Hint:** Start by choosing an explicit isomorphism $T : V \to W$.

(b) (6 pts) Let V be a finite-dimensional vector space, and let $n = \dim(V)$. Prove that for any integer k between 0 and $\dim(V)$ there is a natural **injective** map $\phi : Sub_k(V) \to Sub_{n-k}(V^*)$ where V^* is the dual space. **Hint:** While such ϕ is in principle non-unique, there is a very natural choice for it. You have seen this map before (though you probably did not think of it as a map).

6. Let p be a prime, $\mathbb{Z}_p$ the field of congruence classes $\mod p$ and V a vector space over $\mathbb{Z}_p$ with $\dim(V) = n < \infty$.

(a) (4 pts) Prove that $|Sub_k(V)| = |Sub_{n-k}(V)|$ for all $0 \leq k \leq n$.

(b) (6 pts) Assume that $n = 3$. Find the total number of subspaces of V (of all possible dimensions). Explain your argument in detail.