Homework #9. Due Thursday, November 3rd, in class

Reading:

1. For this homework assignment: § 7.1.

2. For next week's classes: read § 7.2 and the end of § 7.1, go over § 7.3.

HOMEWORK POLICY: In this homework all quiz problems may be discussed with others (following the previously stated rules for QD problems).

Problem 1: Q Prove part (b)(ii) of Theorem 17.3, left in class as an exercise (its statement is recalled below). Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$, and assume that $\chi_T(x)$ splits. Fix $\lambda \in Spec(T)$, and let $K_\lambda = \cup_{n=0}^{\infty} \mathrm{Ker}((T - \lambda I)^n)$ be the corresponding generalized eigenspace. As proved in Theorem 17.3(a), there exists a T-invariant subspace M_λ of V s.t.

$$V = K_\lambda \oplus M_\lambda.$$

Let $R = T_{|K_\lambda} \in \mathcal{L}(K_\lambda)$ and $S = T_{|M_\lambda} \in \mathcal{L}(M_\lambda)$. Prove that $\chi_S(x)$ is not divisible by $x - \lambda$ (equivalently that $\lambda \notin Spec(S)$). **Hint:** Use the fact that the sum $K_\lambda + M_\lambda$ is direct.

Problem 2: Q This problem strengthens the statement of Problem 2(b) in HW#7. Let V be a finite-dimensional vector space, let $T \in \mathcal{L}(V)$ and $n = \dim(V)$.

(a) Let $i \in \mathbb{N}$ be s.t. $\mathrm{Ker}(T^{i+1}) = \mathrm{Ker}(T^i)$. Prove that $\mathrm{Ker}(T^m) = \mathrm{Ker}(T^i)$ for all $m > i$. **Hint:** It is enough to show that $\mathrm{Ker}(T^{i+2}) = \mathrm{Ker}(T^{i+1})$.

(b) Prove that $\cup_{i=0}^{\infty} \mathrm{Ker}(T^i) = \mathrm{Ker}(T^n)$.

(c) Deduce from (b) that for any $\lambda \in Spec(T)$ the generalized eigenspace $K_\lambda = \mathrm{Ker}((T - \lambda I)^n)$.

Problem 3: Let $A \in Mat_{n \times n}(F)$.

(a) Suppose that A is in JCF. As usual, for each $\lambda \in Spec(A)$, denote by m_λ the multiplicity of λ with respect to A. Prove that m_λ is equal to the sum of sizes of all Jordan blocks of A corresponding to λ. **Note:** This result follows from the proof of existence of JCF, but it can be easily obtained by direct computation.

(b) **Q** Suppose that $F = \mathbb{R}$ and $\chi_A(x) = (x-4)(x-5)^2(x-6)^3$. Write down all possible Jordan canonical forms of A, up to equivalence (two Jordan canonical forms are called equivalent if they can be obtained from each other by permutation of blocks).

Problem 4: The goal of this problem is to deduce Cayley-Hamilton theorem from the existence of Jordan Canonical Form (JCF). Note that in our textbook Cayley-Hamilton theorem is used in the proof of the existence of JCF, but the proof of the existence of JCF given in class (which will be completed on Tuesday, Nov 1st) is independent of the Cayley-Hamilton theorem.

(a) Prove the following formula for multiplying 2×2 block matrices (where diagonal blocks are square matrices). Let $A_1, A_2 \in Mat_{n\times n}(F)$ be 2×2 block matrices:

$$A_1 = \begin{pmatrix} B_1 & C_1 \\ D_1 & E_1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} B_2 & C_2 \\ D_2 & E_2 \end{pmatrix}.$$

Assume that B_1 and B_2 have the same size (hence so do E_1 and E_2), that is, $B_1, B_2 \in Mat_{k\times k}(F)$ for some $0 < k < n$ (and hence $E_1, E_2 \in Mat_{k\times k}(F)$). Prove that

$$A_1 A_2 = \begin{pmatrix} B_1B_2 + C_1D_2 & B_1C_2 + C_1E_2 \\ D_1B_2 + E_1D_2 & D_1C_2 + E_1E_2 \end{pmatrix}.$$

Note: I do not of any proof except direct computation, which is straightforward, but may require rather cumbersome notations.

(b) Use induction to generalize (a) to the case of $n \times n$ block matrices.

(c) **Q** Prove Cayley-Hamilton theorem for matrices: Let $A \in Mat_{n\times n}(F)$. Then the characteristic polynomial $\chi_A(x)$ vanishes at A, that is, $\chi_A(A) = 0$. **Hint:** First use the formula from (b) and Problem 3(a) to prove the theorem when A is in JCF. Then deduce the theorem for a general A (using existence of JCF for matrices). Note that we do not assume that $\chi_A(x)$ splits, but this does not cause any problems. Why?

(d) Deduce Cayley-Hamilton theorem for linear transformations: if V is a finite-dimensional vector space and $T \in \mathcal{L}(V)$, then $\chi_T(T) = 0$.

Problem 5: Read (and understand) the subsection on "Invariant Subspaces and Direct Sums" (in § 5.4, pp. 318–321).

Problem 6: Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$ a **nilpotent** linear map. In Lecture 18 we

(a) proved that V has a Jordan basis β for T (that is, a basis s.t. $[T]_\beta$ is in JCF) and

(b) observed that an ordered basis β of V is a Jordan basis $\iff$ β is an (ordered) union of nilpotent T-cycles.

The goal of this problem is to justify two very explicit algorithms for constructing a Jordan basis β in the case $\dim(V) = 3$.
So, below we assume that $\dim(V) = 3$, $T \in \mathcal{L}(V)$ is nilpotent and β is a Jordan basis for T. By a cycle we mean a nilpotent T-cycle.

(a) Prove that

(i) If $rk(T) = 0$ (that is, $T = 0$), β consists of 3 cycles of length 1 and so $[T]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(ii) If $rk(T) = 1$, β consists of a cycle of length 2 and a cycle of length 1 and so $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (assuming β is ordered so that length 2 cycle appears before length 1 cycle).

(iii) If $rk(T) = 2$, β consists of a single cycle of length 3 and so $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(iv) $rk(T)$ cannot equal 3.

(b) **Q** Prove that $T^2 \neq 0$ if and only if $rk(T) = 2$.

(c) Note that if $rk(T) = 0$, any basis β of V is a Jordan basis. Justify the following algorithm for finding β, that is, prove that this algorithm will always produce a Jordan basis, regardless of the choices made. (Note: this algorithm essentially follows from the proof of Theorem 18.4 in class)

(i) Assume that $rk(T) = 1$. Take any nonzero $v \in \mathrm{Im}(T)$ and find $y \in T$ s.t. $T(y) = v$. Then $v \in \mathrm{Ker}(T)$ (since $T^2 = 0$ by (b)) and $\dim(\mathrm{Ker}(T)) = 3 - rk(T) = 2$, so we can find $z \in \mathrm{Ker}(T)$ s.t. $\{v, z\}$ is a basis for $\mathrm{Ker}(T)$. Then $\{v, y, z\}$ (in this order) is a Jordan basis.

(ii) Assume that $rk(T) = 2$. Take any nonzero $v \in \mathrm{Im}(T^2)$ (this is possible by (b)), then find $y_1 \in V$ s.t. $T(y_1) = v$ and then $y_2 \in V$

s.t. $T(y_2) = y_1$ (such y_1 and y_2 exist since $v \in \operatorname{Im}(T^2)$). Then $\{v, y_1, y_2\}$ is a Jordan basis.

(d) Now justify a slightly different algorithm for finding a Jordan basis for T.

(i) Assume that $rk(T) = 1$. Take any $v \notin \operatorname{Ker}(T)$ and let $w = T(v)$. Then $w \in \operatorname{Ker}(T)$ since $T^2 = 0$ by (b). Take any $z \in \operatorname{Ker}(T)$ s.t. $\{w, z\}$ is a basis for $\operatorname{Ker}(T)$. Then $\{w, v, z\}$ is a Jordan basis.

(ii) Assume that $rk(T) = 2$. Since $T^2 \neq 0$ by (b), we can find $v \notin \operatorname{Ker}(T^2)$. Then $\{T^2(v), T(v), v\}$ is a Jordan basis.

(e) State the versions of (a)-(d) for the analogous problem with $\dim(V) = 2$.

(f) **Q** For each of the following matrices $A \in Mat_{3\times 3}(\mathbb{R})$ prove that $T = L_A$ is nilpotent and find $JCF(T)$ and a Jordan basis for T (you may use the algorithm from (c) or (d) or any other algorithm, but in the latter case it needs to be justified).

$$\text{(i) } A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & -2 & 1 \\ 2 & -2 & 0 \end{pmatrix} \quad \text{and} \quad \text{(ii) } A = \begin{pmatrix} 2 & -2 & 6 \\ -1 & 1 & -3 \\ -1 & 1 & -3 \end{pmatrix}$$

Problem 7 Q: Find the Jordan canonical form and a Jordan basis for the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \end{pmatrix}.$$

(This is the same as finding JCF and Jordan basis for $T = L_A$). The general algorithm for computing JCF if given below:

Let V be a finite-dimensional vector space over a field F and let $T \in \mathcal{L}(V)$. We assume that $\chi_T(x)$ splits over F.

Step 1: Compute the eigenvalues of T, call them $\lambda_1, \ldots, \lambda_s$.

Step 2: For each $\lambda \in \mathit{Spec}(T)$ compute the generalized eigenspace K_λ. This can be done by consecutively computing $\operatorname{Ker}(T - \lambda I)$, $\operatorname{Ker}(T - \lambda I)^2$ etc. and stopping at the first i s.t. $\dim \operatorname{Ker}(T - \lambda I)^i = m_\lambda(T)$.

Step 3: For each $\lambda \in \mathit{Spec}(T)$ denote by T_λ the restriction of T to K_λ. Find a Jordan basis for each T_λ, that is, find a basis β_λ for K_λ s.t. $[T_\lambda]_{\beta_\lambda}$ is in JCF. This can be done essentially by the algorithm from the proof of Theorem 18.4 or (in the case $m_\lambda \leq 3$) by variations of that algorithm described in Problem 6. Indeed, by definition of K_λ, the map $T_\lambda - \lambda I \in \mathcal{L}(K_\lambda)$ is nilpotent, so Theorem 18.4 provides an algorithm for finding a basis β_λ of K_λ s.t. $[T_\lambda - \lambda I]_{\beta_\lambda}$ is in JCF with all Jordan blocks having 0's on the diagonal. But $[T_\lambda - \lambda I]_{\beta_\lambda} = [T_\lambda]_{\beta_\lambda} + \lambda [I]_{\beta_\lambda} = [T_\lambda]_{\beta_\lambda} + \lambda I_{m_\lambda}$, and by adding the scalar matrix λI_{m_λ} to a matrix in JCF with 0's on the diagonal, we clearly get a matrix in JCF with λ's on the diagonal. Thus, β_λ is a Jordan basis for T_λ.

Note that to implement the above procedure in practice, say, if we use one of the algorithms from Problem 6, one needs to compute $rk((T_\lambda - \lambda I)^i)$ and $\mathrm{Ker}((T_\lambda - \lambda I)^i)$ for various $i \in \mathbb{N}$. All of this can be done without computing the restriction map T_λ explicitly. Indeed, one can show (check these !!!) that $\mathrm{Ker}((T_\lambda - \lambda I)^i) = \mathrm{Ker}((T - \lambda I)^i)$ and thus $rk((T_\lambda - \lambda I)^i) = rk((T - \lambda I)^i) - \dim V + m_\lambda$ by rank-nullity.

Step 4: Now take the ordered union $\beta_{\lambda_1} \cup \ldots \cup \beta_{\lambda_s}$ of the bases found in Step 3 for each $\lambda_i \in Spec(T)$. This union is a Jordan basis for T (this claim will be justified at the beginning of Lecture 19). To determine $JCF(T)$, we go back to Step 3. By construction, each β_λ is a union of nilpotent $(T - \lambda I)$-cycles. Each nilpotent $(T - \lambda I)$-cycle will yield a Jordan block corresponding to λ in $JCF(T)$ with the size of the block equal to the length of the cycle.