**Homework #8. Due Thursday, October 27th, in class**

**Reading:**

1. For this homework assignment: § 5.2.

2. For next week's classes: read § 7.1, go over § 5.4.

**HOMEWORK POLICY:** In this homework all quiz problems may be discussed with others (following the previously stated rules for QD problems).

**Problem 1: Q** Let $V$ be a vector space over a field $F$. A linear map $T \in \mathcal{L}(V)$ is called **nilpotent** if $T^n = 0$ for some $n \in \mathbb{N}$. Prove that if $T$ is nilpotent, then for any nonzero $\lambda \in F$, the map $T - \lambda I$ is invertible and

$$(T - \lambda I)^{-1} = -\lambda^{-1} \sum_{k=0}^{n-1} (\lambda^{-1} T)^k$$

**Problem 2:** Let $V$ be a finite-dimensional vector space and $W_1, W_2, \ldots, W_k$ subspaces of $V$.

(a) Suppose that $V = W_1 + \ldots + W_k$. Prove that $\sum_{i=1}^k \dim W_i \geq \dim V$ and equality holds $\iff$ the sum $W_1 + \ldots + W_k$ is direct.

(b) **Q** Now suppose that the sum $W_1 + \ldots + W_k$ is direct. Prove that $\sum_{i=1}^k \dim W_i \leq \dim V$ and equality holds $\iff$ $V = W_1 + \ldots + W_k$.

(c) **Q** Let $T \in \mathcal{L}(V)$ and $Spec(T) = \{\lambda_1, \ldots, \lambda_k\}$. Use (b) and Theorems 16.3, 16.4 from class to prove that $T$ is diagonalizable $\iff$ $V = \oplus_{i=1}^k E_{\lambda_i}(T)$.

In parts (a) and (b) give a precise reference for any assertion dealing with direct sums (**Hint:** all relevant properties of direct sums can be easily deduced from Theorem 5.10 in the book).

**Problem 3:** Let $V$ be a vector space and $\beta$ a basis of $V$. Suppose that $\beta_1, \ldots, \beta_k$ are subsets of $\beta$ s.t. $\beta$ is a disjoint union of $\beta_1, \ldots, \beta_k$ (that is, $\beta = \beta_1 \cup \ldots \cup \beta_k$ and $\beta_i \cap \beta_j = \{0\}$ for $i \neq j$).

(a) Prove that $V = \text{Span}(\beta_1) \oplus \ldots \oplus \text{Span}(\beta_k)$.

(b) Deduce that if $\dim V < \infty$ and $\beta = \{v_1, \ldots, v_k\}$, then $V = \text{Span}(v_1) \oplus \ldots \oplus \text{Span}(v_k)$.

**Problem 4: Q** Let $V$ be a finite-dimensional vector space, $n = \dim V$, and let $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $T$ listed with multiplicities (that is, each $\lambda \in Spec(T)$ appears on the list $m_\lambda(T)$ times). Prove that

$$\sum_{i=1}^{n} \lambda_i = \mathrm{tr}(T) \quad \text{and} \quad \prod_{i=1}^{n} \lambda_i = \det(T). \qquad (***)$$

Recall that by definition $\mathrm{tr}(T) = \mathrm{tr}([T]_\beta)$ and $\det(T) = \det([T]_\beta)$ where $\beta$ is any basis of $V$. **Hint:** Relate all four expressions in (***) to the characteristic polynomial $\chi_T(x) = \det(T - xI)$.

**Problem 5: Q** In each of the following examples determine whether the matrix $A$ is diagonalizable over $\mathbb{R}$. If $A$ is diagonalizable, find an invertible matrix $Q$ and a diagonal matrix $D$ s.t. $Q^{-1}AQ = D$.

$$\text{(a)} \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{(b)} \quad A = \begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$$

**Problem 6: Q** Let $V$ be a finite-dimensional vector space and $T \in \mathcal{L}(V)$.

(a) Let $\lambda$ be an eigenvalue of $T$. Prove that any subspace of $E_\lambda(T)$ is $T$-invariant.

(b) Prove that if $U$ and $W$ are $T$-invariant subspaces of $V$, then their sum $U+W$ is also $T$-invariant. Generalize to the sum of any finite collection of subspaces.

(c) Suppose that $Spec(T) = \{\lambda_1, \ldots, \lambda_k\}$. Let $W$ be a subspace of $V$ s.t.

$$W = \sum_{i=1}^{k} (W \cap E_{\lambda_i}(T))$$

(Note that this sum direct since $W \cap E_{\lambda_i}(T) \subseteq E_{\lambda_i}(T)$ and the sum $\sum_{i=1}^{k} E_{\lambda_i}(T)$ is direct). Use (a) and (b) to prove that $W$ is $T$-invariant.

In parts (d) and (e) assume that $T$ is diagonalizable.

(d) Prove the converse of (c): If $W$ is a $T$-invariant subspace of $V$, then $W = \sum_{i=1}^{k}(W \cap E_{\lambda_i}(T))$. **Hint:** Since $T$ is diagonalizable, $V = \oplus_{i=1}^{k} E_{\lambda_i}(T)$ by Problem 2(c), so any $w \in W$ can be written as $w = v_1 + \ldots + v_k$ with $v_i \in E_{\lambda_i}(T)$. Use "Vandermonde argument" to prove that each $v_i \in W$.

(e) Use (a), (b) and (d) to prove that every $T$-invariant subspace of $W$ has a $T$-invariant complement.

**Problem 7:** Let $A \in Mat_{n\times n}(F)$ and $A^t$ its transpose.

(a) Prove that $Spec(A) = Spec(A^t)$ and $\dim E_\lambda(A) = \dim E_\lambda(A^t)$ for each $\lambda \in Spec(A)$.

(b) Prove that $A$ is diagonalizable $\iff$ $A^t$ is diagonalizable using (a)

(c) Now prove that $A$ is diagonalizable $\iff$ $A^t$ is diagonalizable directly from definition (without using (a)).

**Problem 8:** Given a field $F$, two matrices $A, B \in Mat_{n\times n}(F)$ are said to be simultaneously diagonalizable if there is an invertible matrix $Q \in Mat_{n\times n}(F)$ s.t. $Q^{-1}AQ$ and $Q^{-1}BQ$ are both diagonal. Given a finite-dimensional vector space $V$, two linear maps $T, S \in \mathcal{L}(V)$ are said to be simultaneously diagonalizable if there exists a basis $\beta$ of $V$ s.t. $[T]_\beta$ and $[S]_\beta$ are both diagonal.

(a) Let $T, S \in \mathcal{L}(V)$ and $\gamma$ a basis of $V$. Prove that $T$ and $S$ are simultaneously diagonalizable $\iff$ $[T]_\gamma$ and $[S]_\gamma$ are simultaneously diagonalizable.

(b) Prove that if $T, S \in \mathcal{L}(V)$ are simultaneously diagonalizable, then $T$ and $S$ commute with each other, that is, $TS = ST$. State and prove the analogous result about matrices.