**Homework #7. Due Thursday, October 20th, in class**

**Reading:**

1. For this homework assignment: § 5.1, parts of 5.2.
2. For next week's class: § 5.1 and 5.2.

**HOMEWORK POLICY:** In this homework all quiz problems may be discussed with others (following the previously stated rules for QD problems).

**Problem 1:** Let $F$ be a field, $n$ a positive integer, and let $A = (a_{ij}) \in Mat_{n\times n}(F)$.

(a) Prove that $A$ is invertible if and only if its columns form a basis of $F^n$.

(b) Let $V$ be a vector space over $F$ with $\dim V = n < \infty$, and let $\beta = \{v_1, \ldots, v_n\}$ be a basis of $V$. Define the vectors $v'_1, \ldots, v'_n \in V$ by $v'_j = \sum_{i=1}^{n} a_{ij} v_i$. Prove that $\{v'_1, \ldots, v'_n\}$ is a basis of $V$ if and only if $A$ is invertible. **Hint:** Use (a) and the fact that isomorphisms of vector spaces send bases to bases.

(c) Deduce from (b) that for any invertible matrix $A \in Mat_{n\times n}(F)$ there is an ordered basis $\beta'$ of $V$ s.t. $A = [I]_{\beta'}^{\beta}$ (where $I : V \to V$ is the identity map).

**Problem 2: Q** Let $V$ be a vector space and $T : V \to V$ a linear map. In parts (b)–(d) assume that $\dim V < \infty$.

(a) Prove that $\mathrm{Ker}(T^n) \subseteq \mathrm{Ker}(T^{n+1})$ for each $n \geq 1$.

(b) By (a) we get an ascending chain of subspaces of $V$:

$$\mathrm{Ker}(T) \subseteq \mathrm{Ker}(T^2) \subseteq \ldots.$$

Prove that this chain stabilizes after finitely many steps, that is, there is $N \in \mathbb{N}$ s.t. $\mathrm{Ker}(T^n) = \mathrm{Ker}(T^N)$ for all $n \geq N$.

(c) Assume that $\mathrm{Ker}(T) \cap \mathrm{Im}(T) \neq \{0\}$. Prove that $\mathrm{Ker}(T^2) \neq \mathrm{Ker}(T)$.

(d) Use (b) and (c) to show that there exists $n \in \mathbb{N}$ s.t. $\mathrm{Ker}(T^n) \cap \mathrm{Im}(T^n) = \{0\}$, and therefore $V = \mathrm{Ker}(T^n) \oplus \mathrm{Im}(T^n)$ by Homework#3.

**Problem 3:** READ and UNDERSTAND the section on direct sums of several subspaces in § 5.2 (pp. 274-278). This material will be frequently used in the next few lectures.

**Problem 4:** **Q** Let $V$ be a finite-dimensional vector space over a field $F$, $n = \dim(V)$ and $T : V \to V$ a linear map. Let $1 \leq k \leq n-1$. Prove that the following are equivalent:

(i) $V = U \oplus W$ where $\dim U = k$ and both $U$ and $W$ are $T$-invariant

(ii) There is an ordered basis $\beta$ of $V$ s.t.

$$[T]_\beta = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

for some $B \in Mat_{k\times k}(F)$ and $C \in Mat_{(n-k)\times(n-k)}(F)$.

**Note:** This problem is quite similar to Problem 3 on HW#4 (the solution for which is posted online).

**Problem 5:** **Q**

(a) Prove that $tr(AB) = tr(BA)$ for any $A, B \in Mat_{n\times n}(F)$ (recall that $tr$ denotes the trace, the sum of diagonal entries).

(b) Prove that if $A, C \in Mat_{n\times n}(F)$ are similar (that is, $C = Q^{-1}AQ$ for some invertible $Q \in Mat_{n\times n}(F)$), then $tr(A) = tr(C)$.

(c) Use (b) to define $tr(T)$ for a linear transformation $T : V \to V$, where $\dim(V) < \infty$

**Problem 6:** Problem 3(a)(d) after § 5.1 (page 257).

**Problem 7:** Given a matrix $A = (a_{ij}) \in Mat_{n\times n}(F)$, its characteristic polynomial $\chi_A(t)$ is defined by $\chi_A(t) = \det(tI_n - A)$ (note that in the book the characteristic polynomial is defined as $\det(A - tI_n)$ which is the same thing up to the factor of $(-1)^n$).

(a) Prove that $\chi_A(t)$ is a monic polynomial of degree $n$ in $t$ ('monic' means that the leading coefficient is equal to 1), so that $\chi_A(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_0$ for some $a_0, \ldots, a_{n-1} \in F$. Moreover, show that $\chi_A(t) = (t - a_{11}) \ldots (t - a_{nn}) + p(t)$ where $\deg(p(t)) \leq n-2$.

(b)**Q** Prove that in the notations of part (a), $a_{n-1} = -\mathrm{tr}(A)$ and $a_0 = (-1)^n \det(A)$.

**Problem 8: Q** Let $V$ be a finite-dimensional vector space over a field $F$ and $T : V \to V$ a linear map. Let $g(x)$ be a polynomial with coefficients in $F$. Note that since the set of linear maps from $V$ to $V$ is closed under addition, multiplication and also multiplication by scalars from $F$, there is a well-defined linear map $g(T) : V \to V$ (e.g. if $g(x) = x^2 + 2x + 3$, then $g(T) = T^2 + 2T + 3I$ where $I$ is the identity map). Suppose that $\lambda$ is an eigenvalue of $T$ and $v \in V$ is an eigenvector of $T$ corresponding $\lambda$. Prove that $v$ is also an eigenvector of $g(T)$ corresponding to $g(\lambda)$ (so in particular, $g(\lambda)$ is an eigenvalue of $g(T)$).

**Problem 9 (bonus):** Let $A, B \in Mat_{n\times n}(F)$ be s.t. the matrix $I_n + AB$ is invertible. Prove that $I_n + BA$ is also invertible. **Hint:** Use the fact that $C \in Mat_{n\times n}(F)$ is non-invertible $\iff$ the equation $Cu = 0$ has a non-trivial solution $u \in F^n$.