Homework #6. Solutions to selected problems.

Problem 2: In parts (a) and (b) let U, V, W be finite-dimensional vector spaces and $T : U \to V$ and $S : V \to W$ linear maps.

(a) Prove that
$$\text{null}(ST) \leq \text{null}(S) + \text{null}(T)$$
(recall that $\text{null}(R) = \dim(\text{Ker}(R))$ for a linear map R). **Hint:** You can nicely apply the result of one of the problems from the first midterm.

(b) Prove that $\text{rk}(ST) \geq \text{rk}(S) + \text{rk}(T) - \dim(V)$.

(c) State and prove the analogue of (b) dealing with ranks of matrices.

Solution: (a) Note that

$$\text{Ker}(ST) = \{v \in V : ST(v) = 0\} = \{v \in V : S(T(v)) = 0\} = \\ \{v \in V : T(v) \in \text{Ker}(S)\} = T^{-1}(\text{Ker}(S)).$$

Thus, $\text{null}(ST) = \dim(\text{Ker}(ST)) = \dim T^{-1}(\text{Ker}(S))$. On the other hand, by Problem 2 in Midterm 1,

$$\dim(T^{-1}(\text{Ker}(S))) \leq \dim(\text{Ker}(S)) + \dim(\text{Ker}(T)) = \text{null}(S) + \text{null}(T).$$

(b) This follows directly from (a) and rank-nullity theorem.

(c) **Statement:** Let F be a field, $A \in Mat_{m\times n}(F)$ and $B \in Mat_{n\times p}(F)$. Then $\text{rk}(AB) = \text{rk}(A) + \text{rk}(B) - n$.

Proof: Let $U = F^p$, $V = F^n$, $W = F^m$. Let $S = L_A : V \to W$ and $T = L_B : U \to V$. Then $ST(v) = L_A(L_B(v)) = AB \cdot v = L_{AB}(v)$, so $ST = L_{AB}$. Thus $\text{rk}(L_{AB}) \geq \text{rk}(L_A) + \text{rk}(L_B) - n$ by (b). Since $\text{rk}(L_A) = \text{rk}(A)$, $\text{rk}(L_B) = \text{rk}(B)$ and $\text{rk}(L_{AB}) = \text{rk}(AB)$, we obtain that $\text{rk}(AB) \geq \text{rk}(A) + \text{rk}(B) - n$, as desired.

Problem 3: Let $A \in Mat_{m\times n}(F)$, $B \in Mat_{n\times p}(F)$, and suppose that $\text{rk}(A) = m$ and $\text{rk}(B) = n$. Determine $rk(AB)$ and prove your answer. **Hint:** Choose vector space U, V, W with $\dim(U) = p$, $\dim(V) = n$ and $\dim(W) = m$, bases α of U, β of V and γ of W, and let $T : U \to V$ and

$S : V \to W$ be the unique linear maps s.t. and $[S]_\beta^\gamma = A$ and $[T]_\alpha^\beta = B$. What can you say about S and T based on what you know about A and B?

Solution 1: As proved in class, $\text{rk}(AB) \leq \min\{\text{rk}(A), \text{rk}(B)\} = \min\{m, n\}$, so in particular, $\text{rk}(AB) \leq m$. On the other hand, by Problem 2(c), $\text{rk}(AB) \geq \text{rk}(A) + \text{rk}(B) - n = m + n - n = m$. Combining the two inequalities, we conclude that $\text{rk}(AB) = m$.

Solution 2: Let T and S be as in the hint. Then $m = \text{rk}(A) = \text{rk}(S) = \dim(\text{Im}(S))$. On the other hand, $\text{Im}(S) \subseteq W$ and $\dim(W) = m$. Thus, we must have $\text{Im}(S) = W$, so $S : V \to W$ is surjective. By the same argument, $T : U \to V$ is surjective.
Since the composition of surjective maps is surjective, $ST : U \to W$ is surjective. Thus $\text{Im}(ST) = W$, so $\text{rk}(ST) = \dim(\text{Im}(ST)) = \dim(W) = m$. Since $AB = [ST]_\alpha^\gamma$, we have $\text{rk}(AB) = \text{rk}(ST) = m$.

Problem 5: Let F be a field. A matrix $A \in Mat_{n\times n}(F)$ is called **skew-symmetric** if $A^t = -A$ (where A^t is the transpose of A).

(a) Prove that if $A \in Mat_{n\times n}(\mathbb{R})$ and n is odd, then $\det(A) = 0$.

(b) Does the assertion of (a) remain valid if $\mathbb{R}$ is replaced by an arbitrary field F? Prove or give a counterexample.

Solution: (a) The matrix $-A$ is obtained from A by consecutively multiplying each row by -1. Each such operation multiplies determinant by -1, so their composition multiplies determinant by $(-1)^n$. Thus, $\det(-A) = (-1)^n \det(A) = -\det(A)$ since n is odd.
Since $A^t = -A$, we get $\det(A) = \det(A^t) = \det(-A) = -\det(A)$. Thus, $2\det(A) = 0$, and dividing both sides by 2, we get $\det(A) = 0$.
(b) The proof in (a) works over a field F whenever one can divide by 2 in F, and this is possible $\iff$ $2 \neq 0$ in F (by definition $2 = 1 + 1$). If $F = \mathbb{Z}_2$ (or more generally if F is any field of characteristic 2), then $2 = 0$, so the proof in (a) does not work over F.
For a specific example, take $F = \mathbb{Z}_2$. Then $A^t = -A$ for any $Mat_{n\times n}(F)$ since $x = -x$ for all $x \in F$. Thus, any matrix $A \in Mat_{n\times n}(F)$ with $\det(A) \neq 0$ (e.g. $A = I_n$) gives a desired counterexample.

Problem 6: Let $A = (a_{ij}) \in Mat_{n\times n}(F)$.

(a) Suppose that A is diagonal, that is, $a_{ij} = 0$ for all $i \neq j$. Prove directly from the definition of determinant given in class that $\det(A) = a_{11}a_{22}\dots a_{nn}$.

(b) Now suppose that A is upper-triangular, that is, $a_{ij} = 0$ for all $i > j$. Again prove directly from the definition of determinant given in class that $\det(A) = a_{11}a_{22}\dots a_{nn}$.

(c) Let A be upper-triangular, and suppose that $a_{kk} = 0$ for some k. Prove without using determinants that $rk(A) < n$. **Hint:** Consider the first k columns of A.

Solution: (a) $\det(A) = \sum_{\sigma \in S_n} (-1)^\sigma m_\sigma$ where $m_\sigma = a_{1\sigma(1)} \dots a_{n\sigma(n)}$. If $\sigma = 12 \dots n$ is the identity permutation, then $(-1)^\sigma m_\sigma = a_{11}a_{22} \dots a_{nn}$. And if σ is any other permutation, then $\sigma(i) \neq i$ for at least one i, whence $m_\sigma = 0$ since A is diagonal. Hence, $\det(A) = a_{11}a_{22} \dots a_{nn}$.

(b) Using the same notation as in (a), we shall show that if $m_\sigma \neq 0$ for some σ, then σ must be the identity permutation (and thus again $\det(A) = a_{11}a_{22} \dots a_{nn}$).

Suppose that $m_\sigma \neq 0$, so that $a_{k\sigma(k)} \neq 0$ for all $1 \leq k \leq n$. Since A is upper-triangular, $a_{n,i} = 0$ for $i \neq n$, and thus we must have $\sigma(n) = n$.

Similarly, since $a_{n-1,i} = 0$ for $i \neq n-1, n$, we must have $\sigma(n-1) = n-1$ or n. Since σ is bijective and we already established that $\sigma(n) = n$, the only possibility is that $\sigma(n-1) = n-1$.

Repeating this argument $n-2$ more times, we conclude that $\sigma(i) = i$ for all $1 \leq i \leq n$.

(c) As usual, we denote the elements of the standard basis of F^n by $e_1, \dots, e_n$. Since A is upper-triangular and $a_{kk} = 0$, the first k columns of A have zeroes in i^{th} row for all $i \geq k$. Thus, the first k columns of A lie in $\text{Span}(e_1, \dots, e_{k-1})$. Thus, if we denote by $CS_k(A)$ the vector space spanned by the first k columns of A, then $\dim(CS_k(A)) \leq \dim \text{Span}(e_1, \dots, e_{k-1}) = k-1$, so $CS_k(A)$ can be spanned by at most $k-1$ elements. Hence $CS(A)$, the entire column space of A, can be spanned by at most $(k-1) + (n-k) = n-1$ elements, so $\text{rk}(A) = \dim(CS(A)) \leq n-1 < n$.