**Homework #6. Due Thursday, October 13th, in class**

**Reading:**

1. For this homework assignment: § 3.1, 3.2, 4.2, 4.3.
2. For next week's class: § 5.1.

**HOMEWORK POLICY:** In this homework all quiz problems may be discussed with others (following the previously stated rules for QD problems).

**Problem 1:** Let $T : V \to W$ and $S : V \to W$ be linear maps, where $V$ and $W$ are finite-dimensional. Prove that $\mathrm{rk}(T + S) \leq \mathrm{rk}(T) + \mathrm{rk}(S)$.

**Problem 2: Q** In parts (a) and (b) let $U, V, W$ be finite-dimensional vector spaces and $T : U \to V$ and $S : V \to W$ linear maps.

(a) Prove that
$$\mathrm{null}(ST) \leq \mathrm{null}(S) + \mathrm{null}(T)$$
(recall that $\mathrm{null}(R) = \dim(\mathrm{Ker}(R))$ for a linear map $R$). **Hint:** You can nicely apply the result of one of the problems from the first midterm.

(b) Prove that $\mathrm{rk}(ST) \geq \mathrm{rk}(S) + \mathrm{rk}(T) - \dim(V)$.

(c) State and prove the analogue of (b) dealing with ranks of matrices.

**Problem 3: Q** Let $A \in Mat_{m\times n}(F)$, $B \in Mat_{n\times p}(F)$, and suppose that $rk(A) = m$ and $rk(B) = n$. Determine $rk(AB)$ and prove your answer. **Hint:** Choose vector space $U, V, W$ with $\dim(U) = p$, $\dim(V) = n$ and $\dim(W) = m$, bases $\alpha$ of $U$, $\beta$ of $V$ and $\gamma$ of $W$, and let $T : U \to V$ and $S : V \to W$ be the unique linear maps s.t. and $[S]_\beta^\gamma = A$ and $[T]_\alpha^\beta = B$. What can you say about $S$ and $T$ based on what you know about $A$ and $B$?

**Problem 4:**

(a) Let $V$ be a finite-dimensional vector space and $T, S : V \to V$ be linear maps. Prove without using determinants that $TS$ is invertible if and only if $T$ and $S$ are both invertible.

(b) Deduce from (a) that if $A, B \in Mat_{n\times n}(F)$ for some field $F$, then $AB$ is invertible if and only if $A$ and $B$ are both invertible.

(c)**Q** Give an example showing that the assertion of (a) may be false if $V$ is infinite-dimensional (and prove that your example has the required properties). **Hint:** It is even possible that $TS = I$, the identity map, while $T$ and $S$ are both non-invertible.

**Problem 5: Q** Let $F$ be a field. A matrix $A \in Mat_{n\times n}(F)$ is called **skew-symmetric** if $A^t = -A$ (where $A^t$ is the tranpose of $A$).

(a) Prove that if $A \in Mat_{n\times n}(\mathbb{R})$ and $n$ is odd, then $\det(A) = 0$.

(b) Does the assertion of (a) remain valid if $\mathbb{R}$ is replaced by an arbitrary field $F$? Prove or give a counterexample.

**Problem 6:** Let $A = (a_{ij}) \in Mat_{n\times n}(F)$.

(a) **Q** Suppose that $A$ is diagonal, that is, $a_{ij} = 0$ for all $i \neq j$. Prove directly from the definition of determinant given in class that $\det(A) = a_{11}a_{22}\ldots a_{nn}$.

(b) **Q** Now suppose that $A$ is upper-triangular, that is, $a_{ij} = 0$ for all $i > j$. Again prove directly from the definition of determinant given in class that $\det(A) = a_{11}a_{22}\ldots a_{nn}$.

(c) **Q** Let $A$ be upper-triangular, and suppose that $a_{kk} = 0$ for some $k$. Prove without using determinants that $rk(A) < n$. **Hint:** Consider the first $k$ columns of $A$.

(d) Now deduce (b) directly from (a), (c) and properties of determinant established in class.

**Problem 7 (bonus):** Suppose that $M = (m_{ij}) \in Mat_{n\times n}(F)$ has block-diagonal form

$$M = \begin{pmatrix} A_{k\times k} & B_{k\times(n-k)} \\ 0_{(n-k)\times k} & C_{(n-k)\times(n-k)}. \end{pmatrix}$$

for some $1 \leq k \leq n-1$. (Equivalently, suppose that $m_{ij} = 0$ whenever $i > k$ and $j \leq k$). Prove that $\det(M) = \det(A)\det(C)$. **Hint:** There are two completely different approaches – using the definition of determinant (this approach involves some computations, but they are not as terrible as it might first appear) or using elementary transformations (this approach involves virtually no computations, but probably you would need to consider two cases depending on whether $A$ and $C$ are both invertible or not).

**Problem 8 Q:** Read about the Vandermonde determinant at

(a) *http://en.wikipedia.org/wiki/Vandermonde_matrix* and

(b) *http://www.proofwiki.org/wiki/Vandermonde_Determinant.*

See also Problem 22 on page 239 after § 4.3.