Solutions to selected problems in homeworks 9-11 (to be continued).

Problem 9.7: Find the Jordan canonical form and a Jordan basis for the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \end{pmatrix}.$$

Solution: *Step 1:* we compute eigenvalues and multiplicities. Using the formula for computing block-upper-triangular or block-lower- triangular matrices we have

$$\det(A - \lambda I) = (2-\lambda)\det\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix} =$$

$$(2-\lambda)(3-\lambda)\det\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = (2-\lambda)^2(3-\lambda)^2 = (\lambda-2)^2(\lambda-3)^2$$

Thus, there are two eigenvalues 2 and 3 and $m_2(A) = m_3(A) = 2$.

Step 2: We compute dot diagrams $D_2(A)$ and $D_3(A)$.

We have $A - 2I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. The last three vectors are easily seen to be linearly independent, and the first column is zero, so $rk(A-2I) = 3$. Therefore, the first row of $D_2(A)$ has $rk((A-2I)^0) - rk((A-2I)^1) = 4-3 = 1$ dot, so $D_2(A)$ has only one column (which must have length 2).

Similar computation shows that the dot diagram $D_3(A)$ also has only one column of length 2. Therefore,

$$JCF(A) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Step 3: We compute the generalized eigenspaces $K_2(A)$ and $K_3(A)$. In fact, for computational purposes it will be more convenient to work with the linear map $T = L_A$, left multiplication by A (of course, $K_2(A) = K_2(T)$ and

$K_3(A) = K_3(T)$). Since the dot diagram $D_2(T)$ has two rows, we have $K_2(T) = \mathrm{Ker}((T-2I)^2)$ and similarly $K_3(T) = \mathrm{Ker}((T-3I)^2)$. To compute these we first determine the action of the linear maps $(T-2I)^2$ and $(T-3I)^2$ on elements of the standard basis e_1, e_2, e_3, e_4.

We have $T-2I : e_1 \mapsto 0, \quad e_2 \mapsto e_1+e_4, \quad e_3 \mapsto e_2+e_3, \quad e_4 \mapsto e_4$. Hence $(T-2I)^2 : e_1 \mapsto 0 \mapsto 0, \quad e_2 \mapsto e_1+e_4 \mapsto e_4, \quad e_3 \mapsto e_2+e_3 \mapsto e_1+e_2+e_3+e_4, \quad e_4 \mapsto e_4 \mapsto e_4$. Clearly, e_1 and $e_2 - e_4$ lie in $\mathrm{Ker}((T-2I)^2)$, and since $\dim \mathrm{Ker}((T-2I)^2) = m_2(T) = 2$, we have $K_2(T) = \mathrm{Ker}((T-2I)^2) = \mathrm{Span}(e_1, e_2 - e_4)$.

Similarly $T-3I : e_1 \mapsto -e_1, \quad e_2 \mapsto e_1 - e_2 + e_4, \quad e_3 \mapsto e_2, \quad e_4 \mapsto 0$, so

$$(T-3I)^2 : e_1 \mapsto e_1, \quad e_2 \mapsto -e_1 - (e_1 - e_2 + e_4) = -2e_1 + e_2 - e_4,$$
$$e_3 \mapsto e_1 - e_2 + e_4, \quad e_4 \mapsto 0$$

Again we note that e_4 and $e_1 + e_2 + e_3$ lie in $\mathrm{Ker}((T-3I)^2)$, and since $\dim \mathrm{Ker}((T-3I)^2) = m_3(T) = 2$, we have $K_3(T) = \mathrm{Ker}((T-3I)^2) = \mathrm{Span}(e_1+e_2+e_3, e_4)$.

Step 4: Finally, we compute a Jordan basis. In Example 2 of Lecture 27 we explicitly discussed an algorithm for computing a Jordan basis for $T_{|K_\lambda}$ in case when the dot diagram $D_\lambda(T)$ is one column of length 3. Clearly, the same argument shows that if the dot diagram is one column of length 2, the following algorithm works: take any $w \in \mathrm{Ker}((T-\lambda I)^2) \setminus \mathrm{Ker}((T-\lambda I))$ and let $v = (T-\lambda I)w$. Then $\{v, w\}$ is a Jordan basis for $T_{|K_\lambda}$

Since both dot diagrams $D_2(T)$ and $D_3(T)$ consist of one column of length 2, we apply the above algorithm first to $\lambda = 2$ and then to $\lambda = 3$ and then take the union of the corresponding bases.

$\lambda = 2$: From the above computation we see that $\mathrm{Ker}(T-2I) = \mathrm{Span}(e_1)$, so we can let $w = e_2 - e_4$. Then $v = (T-2I)(e_2 - e_4) = (e_1 + e_4) - e_4 = e_1$.

$\lambda = 3$: From the above computation we see that $\mathrm{Ker}(T-3I) = \mathrm{Span}(e_4)$, so we can let $w = e_1 + e_2 + e_3$. Then $v = (T-3I)(e_1+e_2+e_3) = (-e_1) + (e_1 - e_2 + e_4) + e_2 = e_4$. So, our final answer for a Jordan basis (which matches the order blocks for the JCF stated in Step 2) is

$$\{e_1, e_2 - e_4, e_4, e_1 + e_2 + e_3\}.$$

Note that the part of the above Jordan basis corresponding to each eigenvalue λ coincides with the basis of the generalized eigenspace $K_\lambda(T)$ found in Step 3. This is a coincidence!!! Note that the vectors e_1 and e_4 (the initial vectors of nilpotent $(T-2I)$- and $(T-3I)$-cycles, respectively), did not come from Step 3, but reappeared in Step 4.

Problem 11.1:

(a) Prove that there exist NO matrix $A \in Mat_{3\times 3}(\mathbb{Q})$ (where $\mathbb{Q}$ denotes rationals) s.t. $A^2 = 5I$. **Hint:** Use minimal polynomials.

(b) Given an example of a matrix $A \in Mat_{3\times 3}(\mathbb{R})$ s.t. $A^2 = 5I$. Then explain where your proof from (a) would break down if $\mathbb{Q}$ is replaced by $\mathbb{R}$.

Solution: (a) First note that the matrix $A = \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} \in Mat_{2\times 2}(\mathbb{Q})$ does satisfy the equation $A^2 = 5I$, so in our proof we have to use the fact that A is 3×3.

Since $A^2 = 5I$, the polynomial $x^2 - 5$ lies in $Ann(A)$ and thus $\mu_A(x)$ divides $x^2 - 5$. Since $x^2 - 5$ is monic and irreducible over $\mathbb{Q}$, we must have $\mu_A(x) = x^2 - 5$.

Now consider $\chi_A(x)$. We know that $\mu_A(x)$ divides $\chi_A(x)$, so

$$\chi_A(x) = \mu_A(x)p(x) \text{ for some polynomial } p(x) \in \mathcal{P}(\mathbb{Q}). \qquad (***)$$

We also know that $\deg(\chi_A(x)) = 3$ and $\deg(\mu_A(x)) = 2$, so $\deg(p(x)) = 3 - 2 = 1$ by (***). Moreover, $\mu_A(x)$ and $\chi_A(x)$ are both monic, so $p(x)$ is also monic, again by (***). Thus, $p(x) \in \mathcal{P}(\mathbb{Q})$ is monic of degree one, so $p(x) = x - \alpha$ for some $\alpha \in \mathbb{Q}$.

This implies that α is a root of $\chi_A(x)$ in $\mathbb{Q}$. On the other hand, by Lecture 21, $\mu_A(x)$ and $\chi_A(x)$ must have the same roots (in any field), so α is also a rational root of $\mu_A(x) = x^2 - 5$. But the only real roots of $x^2 - 5$ are $\pm\sqrt{5} \notin \mathbb{Q}$, so $x^2 - 5$ has no rational roots, which is a contradiction.

(b) The scalar matrix $A = \sqrt{5}\, I = diag(\sqrt{5}, \sqrt{5}, \sqrt{5})$ satisfies $A^2 = 5I$. The proof from (a) does not work over $\mathbb{R}$ since $x^2 - 5$ is reducible over $\mathbb{R}$.

Problem 11.3: Let F be a field, $n \in \mathbb{N}$ and $V = Mat_{n\times n}(F)$. Define the function $H : V \times V \to F$ by $H(A, B) = \mathrm{tr}(AB)$.

(a) Prove that H is a non-degenerate symmetric bilinear form on V.

(b) Assume that $char(F) \neq 2$. Find an H-orthogonal basis of V.

Solution: (a) Using the fact that $\mathrm{tr} : V \to F$ is a linear map (as established in HW#1), we have $H(A+B, C) = \mathrm{tr}((A+B)C) = \mathrm{tr}(AC+BC) = \mathrm{tr}(AC)+ \mathrm{tr}(BC) = H(A, C)+H(B, C)$. Similarly, $H(A, B+C) = H(A, B)+H(A, C)$ and $H(\lambda A, B) = H(A, \lambda B) = \lambda H(A, B)$ for all $\lambda \in F$, so H is bilinear. Since

$\mathrm{tr}(AB) = \mathrm{tr}(BA)$ by HW#7.5(a), H is symmetric. It remains to show that H is non-degenerate.
For any matrix $A \in V$ we have $A = \sum_{i,j=1}^n A_{ij}e_{ij}$ where A_{ij} is the (i,j)-entry of A and, as usual, e_{ij} is the matrix whose (i,j)-entry is equal to 1 and all other entries are equal to 0. Using the formula $e_{ij}e_{kl} = \delta_{jk}e_{il}$ we have

$$H(A, e_{kl}) = \mathrm{tr}(\sum_{i,j=1}^n A_{ij}e_{ij}e_{kl}) = \mathrm{tr}(\sum_{i,j=1}^n A_{ij}\delta_{jk}e_{il}) = \sum_{i,j=1}^n A_{ij}\delta_{jk}\mathrm{tr}(e_{il}) = \sum_{i,j=1}^n A_{ij}\delta_{jk}\delta_{il}.$$

The only nonzero term in this sum comes from $j = k$ and $i = l$ and is equal to A_{lk}, so $H(A, e_{kl}) = A_{lk}$. Thus, if $A \in \mathrm{LKer}(H)$, then $A_{lk} = 0$ for each $1 \le k, l \le n$, so $A = 0$. Thus, H is non-degenerate.

(b) We start with introducing some terminology. Two subspaces U and W of V are called H-orthogonal if $H(u, w) = 0$ for all $u \in U$ and $w \in W$. The following results are straightforward.

(i) Let β_U be a basis of U and β_W a basis of W, and assume that $H(u, w) = 0$ for all $u \in \beta_U$ and $w \in \beta_W$. Then U and W are H-orthogonal.

(ii) Suppose that $V = \oplus_{i=1}^k W_k$, where W_i and W_j are H-orthogonal for $i \neq j$. Let β_i be a basis of W_i for $1 \le i \le k$ and $\beta = \sqcup_{i=1}^k \beta_i$. Let H_i be the restriction of H to W_i. Then

$$[H]_\beta = \oplus_{i=1}^k [H_i]_{\beta_i},$$

that is, the matrix of H with respect to β is block-diagonal with blocks equal to the matrices of H_i with respect to β_i. In particular, this means that if we choose β_i s.t. $[H_i]_{\beta_i}$ is diagonal for each i, then $[H]_\beta$ is also diagonal.

We now return to our problem. Let $V_i = \mathrm{Span}(e_{ii})$, and for $1 \le i < j \le n$ let $V_{ij} = \mathrm{Span}(e_{ij}, e_{ji})$. It is then clear (e.g. by HW#8.3) that

$$V = (\oplus_{i=1}^n V_i) \oplus (\bigoplus_{1\le i<j\le n} V_{ij})$$

Using fact (i) above and computations in (a), it is easy to check that any two (distinct) subspaces on the RHS of the above decomposition are H-orthogonal. Thus, by (ii) if we find bases β_i of V_i and β_{ij} of V_{ij} s.t $[H_i]_{\beta_i}$

and $[H_{ij}]_{\beta_{ij}}$ are all diagonal (where $H_i = H_{|V_i}$ and $H_{ij} = H_{|V_{ij}}$), then $[H]_\beta$ is diagonal for $\beta = (\sqcup\beta_i) \sqcup (\sqcup\beta_{ij})$.

The subspaces V_i are one-dimensional, so $[H_i]_{\beta_i}$ is diagonal for any choice of β_i, and we simply set $\beta_i = \{e_i\}$.

We claim that we can let $\beta_{ij} = \{e_{ij} - e_{ji}, e_{ij} + e_{ji}\}$ for $i < j$. One way to get this basis is to apply Gram-Schmidt orthogonalization from Theorem 25.4 to the bilinear form H_{ij}. But it is also easy to check directly that β_{ij} works. First, we need to check that β_{ij} is a basis of V_{ij}, and this is where the assumption that $char(F) \neq 2$ comes into play (CHECK THIS). Once this is done, we simply verify that $H(e_{ij} - e_{ji}, e_{ij} + e_{ji}) = H(e_{ij}, e_{ij}) + H(e_{ij}, e_{ji}) - H(e_{ji}, e_{ij}) - H(e_{ji}, e_{ji}) = 0 + 1 - 1 - 0 = 0$, so (since H is symmetric), $[H_{ij}]_{\beta_{ij}}$ is diagonal. Thus, the basis

$$\beta = \{e_{ii} : 1 \leq i \leq n\} \sqcup \{e_{ij} - e_{ji}, e_{ij} + e_{ji} : 1 \leq i < j \leq n\}$$

is H-orthogonal.