Solutions to selected homework problems.

Problem 2.1: Let p be any prime and $V = \mathbb{Z}_p^2$, the standard two-dimensional vector space over $\mathbb{Z}_p$. How many ordered bases does V have?

Answer: $(p^2 - 1)(p^2 - p)$.

Solution: First, by Corollary 3.5(c) any basis of V has two elements.

Lemma: *Let $v, w \in V$. Then $\{v, w\}$ is a basis of $V \iff v \neq 0$ and w is not a multiple of v.*

Proof of Lemma: First note that by Corollary 3.5(e) $\{v, w\}$ is a basis $\iff$ $\{v, w\}$ is linearly independent. Thus, replacing the statement of the lemma by contrapositive, we are reduced to proving the following:

$$\{v, w\} \text{ is linearly dependent} \iff v = 0 \text{ or } w \text{ is a multiple of } v.$$

"$\Leftarrow$" If $v = 0$, then $0 = 1 \cdot v + 0 \cdot w$, and if $w = \lambda v$ for some λ, then $0 = 1 \cdot w + (-\lambda) \cdot v$. In either case $\{v, w\}$ is linearly dependent.

"$\Rightarrow$" If $\{v, w\}$ is linearly dependent, there exist $\lambda, \mu \in F$, not both 0 s.t. $\lambda v + \mu w = 0$. If $\mu = 0$, then $\lambda v = 0$ and $\lambda \neq 0$, so $v = \lambda^{-1}(\lambda v) = 0$. And if $\mu \neq 0$, then $w = -\frac{\lambda}{\mu} v$ is a multiple of v. $\square$

By Lemma, to find the number of bases we need to count the number of ordered pairs (v, w) with $v \neq 0$ and w not a multiple of v. The total number of vectors in V is the number of pairs (a, b) with $a, b \in \mathbb{Z}_p$. There are p choices for a and p choices for b (and there are no dependencies between a and b), so $|V| = p^2$. Thus, there are $p^2 - 1$ nonzero vectors in V, so we have $p^2 - 1$ choices for v.

Since $v \neq 0$, it has precisely p multiples (including itself) – indeed, since $|\mathbb{Z}_p| = p$, there are at most p multiples, namely $0 \cdot v, 1 \cdot v, \ldots, (p-1) \cdot v$; on the other hand, all these multiples are distinct: if $\lambda, \mu \in F$ are such that $\lambda v = \mu v$, then $(\lambda - \mu)v = 0$, and if $\lambda \neq \mu$, multiplying by $(\lambda - \mu)^{-1}$, we get $v = 0$, which is contradiction.

So, once v has been chosen there are precisely $p^2 - p$ choices for w. Therefore, the total number of choices for the ordered pair (v, w) is $(p^2 - 1)(p^2 - p)$.

Problem 2.2: Prove Lemma 3.2 from class: If V is a vector space, S a subset of V and $v \in V$, then $Span(S \cup \{v\}) = Span(S) \iff v \in Span(S)$.

Solution: "$\Rightarrow$" Assume that $Span(S \cup \{v\}) = Span(S)$. Since $v \in S \cup \{v\}$

and $T \subseteq Span(T)$ for any set T, we get $v \in Span(S \cup \{v\}) = Span(S)$.

"$\Leftarrow$" Assume that $v \in Span(S)$. Since $S \subseteq Span(S)$, we have $S \cup \{v\} \subseteq Span(S)$ and therefore $Span(S \cup \{v\}) \subseteq Span(Span(S))$ by Theorem 2.1(e). But $Span(Span(S)) = Span(S)$ by Theorem 2.1(d), so $Span(S \cup \{v\}) \subseteq Span(S)$. The opposite inclusion $Span(S) \subseteq Span(S \cup \{v\}$ is clear (again by Theorem 2.1(e) since $S \subseteq S \cup \{v\}$). $\square$

Problem 2.5(b): Let $V = \mathbb{R}^2$, and let U and W be subspaces of V with $\dim(U) = \dim(W) = 1$. Prove that W is a complement of $U \iff W \neq U$.

Solution: "$\Rightarrow$" By contradiction. Suppose that $W = U$. Since we assume that W is a complement of U, we have $U \cap W = \{0\}$ which together with $W = U$ implies that $U = U \cap U = \{0\}$ and similarly $W = \{0\}$. But then $U + W = \{0\} \neq V$, contrary to the assumption that W is a complement of U.

"$\Leftarrow$" Suppose that $W \neq U$. Then at least one of the following holds: W is not contained in U or U is not contained in W. WOLOG assume that W is not contained in U. Then $U \cap W$ is a proper subspace of W, so by Theorem 1.11 (book) $\dim(U \cap W) < \dim(W)$. Since $\dim(W) = 1$, the only possibility is that $\dim(U \cap W) = 0$ which means that $U \cap W = \{0\}$. Also note that $\dim(U) + \dim(W) = 1 + 1 = 2 = \dim(V)$. Hence, by Problem 2.4(c) we conclude that $V = U \oplus W$, so W is a complement of U.

Problem 3.1 For each of the following maps T do the following: Prove that T is linear and find a basis for $\text{Ker}(T)$ and $\text{Im}(T)$.

(a) $T : P_6(\mathbb{R}) \to P_6(\mathbb{R})$ given by $T(f(x)) = f'(x)$

(b) $T : P_6(\mathbb{Z}_3) \to P_6(\mathbb{Z}_3)$ given by $T(f(x)) = f'(x)$ (where as before $\mathbb{Z}_3$ is the field of congruence classes mod 3).

(c) $T : P_3(\mathbb{R}) \to \mathbb{R}$ given by $T(f(x)) = f(2)$, that is, T is the evaluation map at $x = 2$.

(d) $T : P_3(\mathbb{R}) \to P_4(\mathbb{R})$ given by $T(f(x)) = (x+1)p(x)$, that is, T is the multiplication by $x + 1$.

Answer:

(a) $\text{Ker}(T)$ has basis $\{1\}$ and $\text{Im}(T)$ has basis $\{1, x, x^2, x^3, x^4, x^5\}$.

(b) $\text{Ker}(T)$ has basis $\{1, x^3, x^6\}$ and $\text{Im}(T)$ has basis $\{1, x, x^3, x^4\}$.

(c) $\text{Ker}(T)$ has basis $\{x - 2, (x - 2)^2, (x - 2)^3\}$ and $\text{Im}(T)$ has basis $\{1\}$.

(d) $\mathrm{Ker}(T) = \{0\}$, so has the empty set $\emptyset$ as its only basis and $\mathrm{Im}(T)$ has basis $\{(x+1), x(x+1), x^2(x+1), x^3(x+1)\}$.

Of course, the choice of basis is not unique, and in the case of $\mathrm{Ker}(T)$ in (c) and $\mathrm{Im}(T)$ in (d) there is no particularly natural choice of for a basis.
Justifitcaiton for (c) First note that $\mathrm{Im}(T) = \mathbb{R}$ since for any $\alpha \in \mathbb{R}$ there exists $f(x) \in P_3(\mathbb{R})$ s.t. $f(2) = \alpha$ (e.g. the constant polynomial $f(x) = \alpha$). So, $\dim(\mathrm{Im}(T)) = 1$, and by the rank-nullity theorem $\dim(\mathrm{Ker}(T)) = \dim(P_3(\mathbb{R})) - \dim(\mathrm{Im}(T)) = 4 - 1 = 3$.

The polynomials $x-2, (x-2)^2, (x-2)^3$ vanish at 2, so they lie in $\mathrm{Ker}(T)$. They are also linearly independent (e.g. by HW#1.7 since they have distinct degrees), and since there are $3 = \dim(\mathrm{Ker}(T))$ of them, they must form a basis.

Problem 3.6: Let V be a vector space and $T : V \to V$ a linear map. A subspace W of V is called T-*invariant* if $T(W) \subseteq W$ where $T(W) = \{T(w) : w \in W\}$.

(a) Prove that $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ are T-invariant subspaces

(b) Assume that $\dim(V) < \infty$ and W is a T-invariant subspace of V s.t. $V = W \oplus \mathrm{Ker}(T)$. Prove that $W = \mathrm{Im}(T)$. **Hint:** First show that $\mathrm{Im}(T) \subseteq W$.

(c) Give an example with $\dim(V) < \infty$ where the sum $\mathrm{Im}(T) + \mathrm{Ker}(T)$ is NOT direct.

(d) Use (b) and (c) to conclude that a T-invariant subspace may NOT have a T-invariant complement.

Solution: (a) We know that $0 \in \mathrm{Ker}(T)$. Hence for any $v \in \mathrm{Ker}(T)$ we have $T(v) = 0 \in \mathrm{Ker}(T)$, so $\mathrm{Ker}(T)$ is T-invariant. Now take any $w \in \mathrm{Im}(T)$. Since $\mathrm{Im}(T) \subseteq V$, we have $T(w) \in T(\mathrm{Im}(T)) \subseteq T(V) = \mathrm{Im}(T)$, so $\mathrm{Im}(T)$ is T-invariant.

(b) done in class on September 29th

(c) Take any $n \geq 1$ and consider $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ given by $T(f(x)) = f'(x)$. Then any nonzero constant polynomial lies in both $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$, so $\mathrm{Ker}(T) \cap \mathrm{Im}(T) \neq \{0\}$ and thus the sum $\mathrm{Ker}(T) + \mathrm{Im}(T)$ cannot be direct.

(d) Let us take any map $T : V \to W$ where the sum $\mathrm{Ker}(T) + \mathrm{Im}(T)$ is not direct (e.g. take the above map from (c)). We know that $\mathrm{Ker}(T)$ is T-invariant. Suppose that $\mathrm{Ker}(T)$ has a T-invariant complement, that is, there is a T-invariant subspace W s.t. $V = \mathrm{Ker}(T) \oplus W$. Then by (b) $W = \mathrm{Im}(T)$. This contradicts the assumption that the sum $\mathrm{Ker}(T) + \mathrm{Im}(T)$ is not direct.

Problem 3.7: Recall that $\mathfrak{sl}_n(F)$ denotes the space of all $n \times n$ matrices over F with trace 0. In Problem 2 of HW#6 it was proved that $\dim(\mathfrak{sl}_n(F)) = n^2 - 1$ after a considerable amount of work. Now give a short proof of this fact by applying the rank-nullity theorem to a suitable linear map.

Solution: Let $Mat_n(F)$ be the vector space of all $n \times n$ matrices over F, and consider the map $T : Mat_n(F) \to F$ given by $T(A) = \mathrm{tr}(A)$. Then $\mathrm{Ker}(T) = \mathfrak{sl}_n(F)$ (by definition) and $\mathrm{Im}(T) = F$ since any $\alpha \in F$ is the trace of some $A \in Mat_n(F)$ (e.g. $\alpha = \mathrm{tr}(\alpha e_{11})$). So, $\dim(\mathrm{Im}(T)) = 1$ and by the rank-nullity theorem $\dim(\mathrm{Ker}(T)) = \dim(Mat_n(F)) - \dim(\mathrm{Im}(T)) = n^2 - 1$.

Problem 4.3: Let V be a finite-dimensional vector space and $k \leq \dim(V)$ a positive integer. Let $T : V \to V$ be a linear transformation. Prove that the following are equivalent:

(a) There exists a T-invariant subspace W of V with $\dim(W) = k$ (recall that the notion of a T-invariant subspace is defined in Problem#6 of Homework#3).

(b) There exists a basis $\mathcal{B}$ of V s.t. the matrix $[T]_{\mathcal{B}}$ has the block-diagonal form

$$\begin{pmatrix} A_{k\times k} & B_{k\times(n-k)} \\ 0_{(n-k)\times k} & C_{(n-k)\times(n-k)}. \end{pmatrix}$$

where subscrpits indicate matrix sizes and $0_{(n-k)\times k}$ is the $(n-k) \times k$ zero matrix.

Solution: "(b)$\Rightarrow$ (a)" Suppose that $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $[T]_{\mathcal{B}} = (a_{ij})_{1\leq i,j\leq n}$.

By definition of $[T]_{\mathcal{B}}$ we have $T(v_j) = \sum_{i=1}^{n} a_{ij} v_i$ for all $1 \leq j \leq n$. On the other hand, the assumption about the block-diagonal form of $[T]_{\mathcal{B}}$ from (b) implies that $a_{ij} = 0$ for $k+1 \leq i \leq n$ and $1 \leq j \leq k$. This means that

$$T(v_j) = \sum_{i=1}^{k} a_{ij} v_i \text{ for all } 1 \leq j \leq k. \qquad (***)$$

Let $W = Span(v_1, v_2, \ldots, v_k)$; note that $\dim(W) = k$ since $\{v_1, \ldots, v_k\}$ is linearly independent, being a subset of a basis of V. By (***) $T(v_j) \in W$ for all $1 \leq j \leq k$, and since T is linear (and W is a subspace), we conclude that $T(w) \in W$ for all $w \in Span(v_1, v_2, \ldots, v_k) = W$. So, $T(W) \subseteq W$, and thus W is a T-invariant subspace with $\dim(W) = k$.

"(a)$\Rightarrow$ (b)" Choose an ordered basis $\{v_1, \ldots, v_k\}$ of W and extend it to an ordered basis $\{v_1, \ldots, v_n\}$ of V; call the latter basis $\mathcal{B}$. Since W is T-invariant, for each $1 \leq j \leq k$ we have $T(v_j) \in W$, so $T(v_j) = \sum_{i=1}^{k} a_{ij} v_i = \sum_{i=1}^{k} a_{ij} v_i + \sum_{i=k+1}^{n} 0 \cdot v_i$. This implies that the (i, j) entry of $[T]_{\mathcal{B}}$ is equal to 0 whenever $1 \leq j \leq k$ and $k+1 \leq i \leq n$, so $[T]_{\mathcal{B}}$ has the required block-diagonal form.

Problem 4.5: Let V be a finite-dimensional vector space and $T : V \to V$ a linear map. Prove that the following are equivalent:

(i) T is a projection, that is, $T = p_{U,W}$ for some U and W with $U \oplus W = V$ (ntte that $p_{U,W}$ is defined in Problem 4.4).

(ii) $T^2 = T$ (where $T^2 = T \cdot T$ is the composition of T with itself)

(iii) There is an ordered basis β of V s.t. $[T]_\beta = e_{11} + \ldots + e_{kk}$ for some $k \leq \dim V$, that is, $[T]_\beta$ is the diagonal matrix whose first k diagonal entries are equal to 1 and the remaining diagonal entries are equal to 0.

Solution: We'll prove that (i)⇒(iii)⇒(ii)⇒(i).

"(i)⇒(iii)": choose some ordered bases $\{u_1, \ldots, u_k\}$ of U and $\{w_1, \ldots, w_l\}$ of W. We claim that their ordered union $\beta = \{u_1, \ldots, u_k, w_1, \ldots w_l\}$ (with elements of W listed first) is a basis of V – this is not hard to prove directly, but we can also deduce it from previous homework problems. Indeed, by Problem 2.1(d) $Span(\beta) = Span(\{u_1, \ldots, u_k\}) + Span(\{w_1, \ldots, w_l\}) = U + W$, so β spans $U + W = V$. Since $V = U \oplus W$ (the sum is direct), by Problem 2.4, $\dim(V) = \dim(U) + \dim(W) = k + l = |\beta|$, so β is a basis of V by Corollary 3.5(d).

Since $T = p_{U,W}$, we have $T(u_j) = u_j$ for $1 \leq j \leq k$ and $T(w_j) = 0$ for $1 \leq j \leq l$. We conclude that $[T]_\beta$ has 1 as its (j,j)-entry for all $1 \leq j \leq k$ and all other entires are 0. Therefore, $[T]_\beta = e_{11} + \ldots + e_{kk}$.

"(iii)⇒(ii)": Since $[T]_\beta = e_{11} + \ldots + e_{kk}$, by direct computation we have $([T]_\beta)^2 = [T_\beta]$. On the other hand, by Theorem 2.14(book) $([T]_\beta)^2 = [T^2]_\beta$. So, $[T^2]_\beta = [T]_\beta$, and since a linear map is uniquely determined by its matrix with respect to a given a basis, we conclude that $T^2 = T$.

"(ii)⇒(i)": Assume that $T^2 = T$. We claim that

$$V = \mathrm{Ker}(T) \oplus \mathrm{Im}(T).$$

Take any $v \in \mathrm{Ker}(T) \cap \mathrm{Im}(T)$. Then $T(v) = 0$ and $v = T(u)$ for some u, so $v = T(u) = T^2(u) = T(T(u)) = T(v) = 0$. Hence $\mathrm{Ker}(T) \cap \mathrm{Im}(T) = \{0\}$. On the other hand, by the rank-nullity theorem $\dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T)) = \dim(V)$. Combining the two results, we conclude that $V = \mathrm{Ker}(T) \oplus \mathrm{Im}(T)$ by Problem 2.4(c). (It is also not hard to show directly that $V = \mathrm{Ker}(T) + \mathrm{Im}(T)$: indeed, any $v \in V$ can be written as $v = (v - T(v)) + T(v)$ and $v - T(v) \in \mathrm{Ker}(T)$ for $T(v - T(v)) = T(v) - T^2(v) = 0$).

Now let $U = \mathrm{Im}(T)$ and $W = \mathrm{Ker}(T)$. Then $T(w) = 0$ for all $w \in W$ and $T(u) = u$ for all $u \in U$ (for any $u \in U$ can be written as $u = T(z)$ for some z and $T(u) = T(T(z)) = T(z) = u$). Therefore, $T = p_{U,W}$ by definition.

Problem 5.6: Prove Proposition 10.4: Let V be a finite-dimensional vector space and W a subspace of V. Then

$$\dim(W) + \dim(Ann(W)) = \dim(V).$$

See Problem 14 in § 2.6 for a hint.

Solution: Let $n = \dim V$ and $m = \dim W$. Following the hint in the book, choose a basis $\{v_1, \ldots, v_m\}$ of W and extend it to a basis $\{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ of V. Let $\{v_1^*, \ldots, v_n^*\}$ be the dual basis of V^*, and let

$$B = \{v_{m+1}^*, \ldots, v_n^*\}$$

Let us show that B is a basis of $Ann(W)$ (this would imply that $\dim(Ann(W)) = n - m = \dim V - \dim W$, as desired).

Note that B is linearly independent (being a subset of a basis of V^*), so we only need to check that $Ann(W) = Span(B)$.

Part 1: $Span(B) \subseteq Ann(W)$. First take any element of B, that is, v_i^* with $m+1 \leq i \leq n$. Then $v_i^*(v_j) = 0$ for all $1 \leq j \leq m$ (by definition of dual basis), and by linearity $v_i^*(\lambda_1 v_1 + \ldots + \lambda_m v_m) = 0$ for all $\lambda_1, \ldots, \lambda_m \in F$. So, $v_i^*(w) = 0$ for all $w \in Span(v_1, \ldots, v_m) = W$, so $v_i^* \in Ann(W)$.

Thus, we proved that $B \subseteq Ann(W)$, and since $Ann(W)$ is a subspace (by Problem 5.5), it follows that $Span(B) \subseteq Ann(W)$.

Part 2: $Ann(W) \subseteq Span(B)$. Take any $f \in Ann(W)$. Since $\{v_1^*, \ldots, v_n^*\}$ is a basis of V^*, we can write

$$f = \lambda_1 v_1^* + \ldots \lambda_n v_n^* \text{ for some } \lambda_1, \ldots, \lambda_n \in F. \qquad (***)$$

Since $f \in Ann(W)$, we must have $f(v_i) = 0$ for $1 \leq i \leq m$. Fix such i and evaluate both sides of (***) at v_i. Since $v_k^*(v_i) = 0$ for $k \neq i$ and 1 for $k = i$, we get that $f(v_i) = \lambda_i$ (and recall that $f(v_i) = 0$). So, $\lambda_i = 0$ for $1 \leq i \leq m$, and therefore, $f = \sum_{i=m+1}^{n} \lambda_i v_i^* \in Span(B)$. $\square$