Homework #5. Due Thursday, September 29th, in class

Reading:

1. For this homework assignment: § 2.6.

2. For next week's classes: § 3.1 - 3.3. On Tuesday we will mostly talk about the rank of matrices (3.2) and basic criteria for solvability of the linear matrix equation $Ax = b$. On Thursday no new material will be presented; instead we will go over homework problems from assignments 1-4.

HOMEWORK POLICY: In this homework all quiz problems may be discussed with others (following the previously stated rules for QD problems).

Problem 1: **Q** In each of the following examples you are given a finite-dimensional vector space V and a basis $\beta = \{v_1, \ldots, v_n\}$ of V. Find the dual basis $\beta^* = \{v_1^*, \ldots, v_n^*\}$ and give an explicit formula for each element of β^*. For instance, in part (a) the answer should be given in the form $v_1^*((a, b)) =$ some explicit function of a, b and similarly for v_2^*.

(a) **Q** $V = \mathbb{R}^2$, $\beta = \{(3, 4), (7, 8)\}$

(b) $V = P_1(\mathbb{R})$, $\beta = \{x + 1, x + 2\}$.

Problem 2: **Q** Let $V = P_2(\mathbb{R})$ and consider the elements f_1, f_2, f_3 of V^* given by $f_1(a+bx+cx^2) = a+b$, $f_2(a+bx+cx^2) = b+c$ and $f_3(a+bx+cx^2) = a+c$. Prove that $\{f_1, f_2, f_3\}$ is a basis of V^* and find a basis β of V for which $\{f_1, f_2, f_3\}$ is the dual basis.

Problem 3: Let F be a field and $V = P(\mathbb{R})$, the vector space of all polynomials with coefficients in F. Let $\beta = \{1, x, x^2, \ldots\}$ and define the elements $\{(x^n)^* : n \in \mathbb{Z}_{\geq 0}\}$ of V^* by $((x^n)^*)(x^m) = \delta_{n,m}$ for all $n, m \in \mathbb{Z}_{\geq 0}$. Let $W = Span(\{(x^n)^* : n \in \mathbb{Z}_{\geq 0}\})$.

(a) Let $f \in V^*$. Prove that $f \in W \iff$ there are only finitely many $m \in \mathbb{Z}_{\geq 0}$ for which $f(x^m) \neq 0$.

(b) Use (a) to explicitly construct an element of V^* which does not lie in W.

Problem 4: Fill in the details of the proof of Theorem 10.2 in class (verify all the statement involving linearity).

Problem 5: Recall that if S is a subspace of a vector space V, the **annihilator** of S, denoted by $Ann(S)$ is defined by

$$Ann(S) = \{f \in V^* : f(s) = 0 \text{ for all } s \in S.\}$$

Prove the following statements:

(i) $Ann(S)$ is a subspace for any set S

(ii) $Ann(S_1) \subseteq Ann(S_2)$ whenever $S_2 \subseteq S_1$

(iii)**Q** $Ann(S) = Ann(Span(S))$ for any set S

(iv)**Q** $Ann(S_1 \cup S_2) = Ann(S_1) \cap Ann(S_2)$ for subsets S_1 and S_2

(v) **bonus** Given a subset S of V, define

$$\widetilde{S} = \{v \in V : f(v) = 0 \text{ for all } f \in Ann(S)\}.$$

Prove that $\widetilde{S} = Span(S)$. **Hint:** The inclusion $\widetilde{S} \supseteq Span(S)$ is relatively easy. To prove the opposite inclusion take any $w \in V \setminus Span(S)$ and use Facts A and B (Lectures 9-10) to show that there exists $f \in Ann(S)$ with $f(w) \neq 0$.

(vi)**Q** Assume that $\dim V < \infty$. Prove that $Ann(Ann(S)) = Span(\iota(S))$ for any set S, where $\iota : V \to V^{**}$ is the natural isomorphism from Theorem 10.2. **Hint: Use (v)**.

Problem 6: Q Prove Proposition 10.4: Let V be a finite-dimensional vector space and W a subspace of V. Then

$$\dim(W) + \dim(Ann(W)) = \dim(V).$$

See Problem 14 in § 2.6 for a hint.