Homework #4. Due Thursday, September 22nd, in class

Reading:

1. For this homework assignment: § 2.2-2.5.
2. For next week's classes: § 2.6, 3.1 and 3.2.

HOMEWORK POLICY: The general policy is the same as in Homework#3. In this assignment there are four (4) quiz problems: 2,5,6(a)(b),7. You are allowed to consult with others (following the previously stated rules for QD problems) on (at most) 2 out of 4 problems. The remaining problems should be done individually, and you should indicate on top of your assignment which problems were discussed with others. In problem 5 each implication in the suggested approach counts as one-third of a problem.

Problem 1: Do some standard computational exercises on change of bases at the end of § 2.5 (do as many as you need to get comfortable).

Problem 2: Q Let $V = P_2(\mathbb{R})$ and define the linear map $T : V \to V$ by $T(f(x)) = (x+1)f'(x)$.

(a) Compute $[T]_{\mathcal{B}}$ for the ordered basis $\mathcal{B} = (1, x, x^2)$.

(b) Let $\mathcal{B}' = (1, (x+1), (x+1)^2)$. Compute the matrix $Q = [id_V]^{\mathcal{B}}_{\mathcal{B}'}$ which changes $\mathcal{B}'$-coordinates into $\mathcal{B}$-coorinates and then use the change of basis formula to compute $[T]_{\mathcal{B}'}$.

(c) Now compute $[T]_{\mathcal{B}'}$ directly from the definition of T. **Hint:** If an element $f \in P_2(\mathbb{R})$ is given in the standard form ($f = ax^2 + bx + c$), to compute its $\mathcal{B}'$-coordinates, let $y = x + 1$ (so that $x = y - 1$) and then expand f as a polynomial in y.

Problem 3: Let V be a finite-dimensional vector space and $k \leq \dim(V)$ a positive integer. Let $T : V \to V$ be a linear transformation. Prove that the following are equivalent:

(a) There exists a T-invariant subspace W of V with $\dim(W) = k$ (recall that the notion of a T-invariant subspace is defined in Problem#6 of Homework#3).

(b) There exists a basis $\mathcal{B}$ of V s.t. the matrix $[T]_{\mathcal{B}}$ has the block-diagonal form
$$\begin{pmatrix} A_{k\times k} & B_{k\times(n-k)} \\ 0_{(n-k)\times k} & C_{(n-k)\times(n-k)}. \end{pmatrix}$$
where subscrpits indicate matrix sizes and $0_{(n-k)\times k}$ is the $(n-k)\times k$ zero matrix.

Problem 4: Let V be a vector space and U and W subspaces of V s.t. $V = U \oplus W$. Prove that there exists unique linear map $T : V \to V$ s.t. $T(u) = u$ for all $u \in U$ and $T(w) = 0$ for all $w \in W$. Such map T is called the **projection onto** U **along** W and will be denoted by $p_{U,W}$. By a **projection** we will mean a map $p_{U,W}$ for some U and W as above.

Problem 5: Q Let V be a finite-dimensional vector space and $T : V \to V$ a linear map. Prove that the following are equivalent:

(i) T is a projection, that is, $T = p_{U,W}$ for some U and W with $U \oplus W = V$.

(ii) $T^2 = T$ (where $T^2 = T\cdot T$ is the composition of T with itself)

(iii) There is an ordered basis $\mathcal{B}$ of V s.t. $[T]_{\mathcal{B}} = e_{11} + \ldots + e_{kk}$ for some $k \le \dim V$, that is, $[T]_{\mathcal{B}}$ is the diagonal matrix whose first k diagonal entries are equal to 1 and the remaining diagonal entries are equal to 0.

Hint: Perhaps the shortest way to prove this is via the cycle "(i)$\Rightarrow$ (iii)$\Rightarrow$ (ii)$\Rightarrow$ (i)". The first two implications are relatively easy. To prove "(ii)$\Rightarrow$ (i)" first show that if $T^2 = T$, then $V = \mathrm{Im}(T) \oplus \mathrm{Ker}(T)$.

Problem 6: Let V and W be vector spaces over the same field and $\mathcal{L}(V, W)$ the vector space of all linear maps from V to W.

(a) **Q** Suppose that W is written as an internal direct sum of its subspaces $W_1, \ldots, W_n$:
$$W = \oplus_{i=1}^{n} W_i.$$
(Recall that direct sums of more than two subspaces were defined in Problem 5 of Homework#4). Let $T_1, \ldots, T_n \in \mathcal{L}(V, W)$ be NONZERO maps such that $\mathrm{Im}(T_i) \subseteq W_i$ for $1 \le i \le n$. Prove that $T_1, \ldots, T_n$ are linearly independent (as elements of $\mathcal{L}(V, W)$).

(b) **Q** Now suppose that $V = \oplus_{j=1}^{m} V_j$ and let $S_1, \ldots, S_m \in \mathcal{L}(V, W)$ be NONZERO maps such that $S_i(v) = 0$ whenever $v \in V_j$ for $j \neq i$ (equivalently, $V_j \subseteq \mathrm{Ker}(S_i)$ for all $j \neq i$). Prove that $S_1, \ldots, S_m$ are linearly independent.

(c) State and prove a theorem which provides a natural generalization for both (a) and (b).

Problem 7: Q Let V be vector space and $T : V \to V$ a linear map.

(a) Assume that $\dim(V) < \infty$. Prove that T is bijective $\iff$ T is injective $\iff$ T is surjective. **Hint:** Use the rank-nullity theorem.

(b) Give an example of an infinite-dimensional vector space $T : V \to V$ and linear maps $T_1, T_2 : V \to V$ s.t. T_1 is injective but not surjective and T_2 is surjective but not injective.