Homework #3. Due Thursday, September 15th, in class

Reading:

1. For this homework assignment: § 2.1 and the part of § 2.4 dealing with isomorphisms.

2. For next week's classes: the rest of Chapter 2 up to § 2.5 (inclusive). I hope to send you a more detailed plan over the weekend.

Problems:

NEW HOMEWORK POLICY: In this assignment there are five (5) quiz problems. You are allowed to consult with others (following the rules for QD problems stated in the previous homework) on (at most) 2 out of 5 problems. The remaining problems should be done individually, and you should indicate on top of your assignment which problems were discussed with others. In case of multiple part problems, each part of a problem with n parts counts with weight $1/n$, so for instance you may discuss with others problems 1(c)(d), 6(b)(c) and 7.

Problem 1: Q For each of the following maps T do the following: Prove that T is linear, find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ and then verify the assertion of the rank-nullity theorem. As usual, $P_n(F)$ denotes the space of polynomials of degree $\leq n$ over a field F.

(a) $T : P_6(\mathbb{R}) \to P_6(\mathbb{R})$ given by $T(f(x)) = f'(x)$

(b) $T : P_6(\mathbb{Z}_3) \to P_6(\mathbb{Z}_3)$ given by $T(f(x)) = f'(x)$ (where as before $\mathbb{Z}_3$ is the field of congruence classes mod 3). **WARNING:** The answer in (b) is different from the answer in (a).

(c) $T : P_3(\mathbb{R}) \to \mathbb{R}$ given by $T(f(x)) = f(2)$, that is, T is the evaluation map at $x = 2$.

(d) $T : P_3(\mathbb{R}) \to P_4(\mathbb{R})$ given by $T(f(x)) = (x+1)p(x)$, that is, T is the multiplication by $x + 1$.

Problem 2: This problem illustrates a relationship between the internal and external direct sums (see also Prolems 4 and 5). Let U and W be vector spaces over the same field and let $V = U \oplus W$ be their external direct sum, that is, V is the set of ordered pairs $\{(u, w) : u \in U, w \in W\}$ with componentwise addition and scalar multiplication. Define the subsets $\widetilde{U}$ and $\widetilde{W}$ of V by $\widetilde{U} = \{(u, 0) : u \in U\}$ and $\widetilde{W} = \{(0, w) : u \in U\}$. Prove that

(i) $\widetilde{U}$ and $\widetilde{W}$ are subspaces of V isomorphic to U and W, respectively

(ii) $V = \widetilde{U} \oplus \widetilde{W}$, where now we talk about the internal direct sum.

Conclude that the external direct sum of two spaces U and W is equal to the internal direct sum of isomorphic copies of U and W.

Problem 3: Q Let V be a vector space over a field F and let $V^2 = V \oplus V$ be the external direct sum of two copies of V. Fix $\lambda \in F$ and put $W = \{(v, \lambda v) : v \in V\}$. Prove that W is a subspace of V^2 which is isomorphic to V.

Problem 4: Q Let U and W be subspaces of a finite-dimensional vector space V. Consider the external direct sum $U \oplus W = \{(u, w) : u \in U, w \in W\}$ and the map $T : U \oplus W \to V$ given by $T((u, w)) = u + w$.

(a) Prove that T is linear and use the rank-nullity theorem to prove that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

(recall that this was proved by a much longer argument in HW#2).

(b) Show that if the internal sum $U + W$ is direct, then T establishes an isomorphism between the external and internal direct sums of U and W.

Problem 5: This problem defines the notion of direct sum of more than two spaces. Let $V_1, \ldots, V_n$ be vector spaces over the same field F. The external direct sum $\oplus_{i=1}^{n} V_i$ is defined as the set of n-tuples $\{(v_1, \ldots, v_n) : v_i \in V_i \text{ for } 1 \leq i \leq n\}$ with componentwise addition and scalar multiplication. Now assume that $V_1, \ldots, V_n$ are subspaces of the same space V. Their internal sum $\sum_{i=1}^{n} V_i$ is defined as

$$\sum_{i=1}^{n} V_i = \{v \in V : v = \sum_{i=1}^{n} v_i \text{ for some } v_i \in V_i\}.$$

As in Problem #4, there is a natural linear map $T : \oplus_{i=1}^{n} V_i \to \sum_{i=1}^{n} V_i$ (where the sum on the left is the external direct sum) given by

$$T((v_1, \ldots, v_n)) = \sum_{i=1}^{n} v_i.$$

Prove that the following are equivalent:

(a) T is an isomorphism

(b) $\mathrm{Ker}(T) = \{0\}$

(c) For all $v \in \sum_{i=1}^{n} V_i$ there exist UNIQUE vectors $v_1 \in V_1, \ldots, v_n \in V_n$ s.t. $v = \sum_{i=1}^{n} v_i$

(d) $V_i \cap (\sum_{j \neq i} V_j) = 0$ for all $1 \leq i \leq n$

(e) $V_i \cap (\sum_{j=1}^{i-1} V_j) = 0$ for all $2 \leq i \leq n$

We say that the sum $\sum_{i=1}^{n} V_i$ is direct and write $\oplus_{i=1}^{n} V_i$ if one (hence also all others) of the above five conditions hold.

Problem 6: Q Let V be a vector space and $T : V \to V$ a linear map. A subspace W of V is called T-*invariant* if $T(W) \subseteq W$ where $T(W) = \{T(w) : w \in W\}$.

(a) Prove that $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ are T-invariant subspaces

(b) Assume that $\dim(V) < \infty$ and W is a T-invariant subspace of V s.t. $V = W \oplus \mathrm{Ker}(T)$. Prove that $W = \mathrm{Im}(T)$. **Hint:** First show that $\mathrm{Im}(T) \subseteq W$.

(c) Give an example with $\dim(V) < \infty$ where the sum $\mathrm{Im}(T) + \mathrm{Ker}(T)$ is NOT direct.

(d) Use (b) and (c) to conclude that a T-invariant subspace may NOT have a T-invariant complement.

Problem 7: Q Recall that $\mathfrak{sl}_n(F)$ denotes the space of all $n \times n$ matrices over F with trace 0. In Problem 2 of HW#6 it was proved that $\dim(\mathfrak{sl}_n(F)) = n^2 - 1$ after a considerable amount of work. Now give a short proof of this fact by applying the rank-nullity theorem to a suitable linear map.