Homework #2. Due Thursday, September 8th, in class

Important addition to homework policy: Problems marked with the **QD** symbol (as opposed to **Q** symbol) are quiz problems which you are allowed to discuss with others. However, the discussion should be limited to key ideas and should terminate by the time you start writing your solution.

Reading:

1. For this homework assignment: § 1.5 - 1.6 and Appendix on fields. Make sure to go over the ALL the theorems in 1.5 and 1.6 (there are some important results, e.g., Theorem 1.8, which we did not discuss in class, but which will be frequently used later).

2. For next week's classes: During Lectures 5-8 we will cover (most of) the material in § 2.1-2.5, but in a slightly different order. A more precise plan will be posted on the webpage this weekend. For now you can start reading § 2.1.

Problems:

Problem 1: Q (Problem 5(c) from Homework#1): Let p be any prime and $V = \mathbb{Z}_p^2$, the standard two-dimensional vector space over $\mathbb{Z}_p$. How many ordered bases does V have? Make sure to explain your reasoning. **Hint:** First count how many choices there are for the first basis element and then how many choices for the second element once the first element has been chosen.

Problem 2: Q Prove Lemma 3.2 from class: If V is a vector space, S a subset of V and $v \in V$, then $Span(S \cup \{v\}) = Span(S) \iff v \in Span(S)$. **Hint:** The forward direction ("$\Rightarrow$") is easy. The backward direction ("$\Leftarrow$") can be proved by a direct computation, but one can give an elegant proof using Theorem 2.1(d).

Problem 3: (a) Let V be a vector space, B a basis of V, and let B_1 and B_2 are subsets of B s.t. $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$ (in such case we say that B is a **disjoint union** of B_1 and B_2 and write $B = B_1 \sqcup B_2$). Let $U = Span(B_1)$ and $W = Span(B_2)$. Prove that W is a complement of U in V, that is, $V = U \oplus W$. **Hint:** By definition to prove that $V = U \oplus W$ one needs to show two things: (i) $V = U + W$ and (ii) $U \cap W = \{0\}$. Assertion (i) follows from the fact that B spans V and (ii) from the fact that B is linearly independent.

(b) Use (a) to fill in the missing details in the proof of Proposition 4.3 from class.

Problem 4: Let V be a finite-dimensional vector space and U and W subspaces of V.

(a) Prove that $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$. See Problem 29(a) in § 1.6 for a hint.

(b) Prove that $V = U \oplus W$ if and only if $V = U + W$ and $\dim(U) + \dim(W) = \dim(V)$

Q(c) Prove that $V = U \oplus W$ if and only if $U \cap W = \{0\}$ and $\dim(U) + \dim(W) = \dim(V)$

Problem 5: Let $V = \mathbb{R}^2$.

(a) Let $w = (1,2)$ and $W = Span(w)$. Find infinitely many different complements for W. Can you describe all complements?

Q (b) Let U and W be subspaces of V with $\dim(U) = \dim(W) = 1$. Prove that W is a complement of $U \iff W \neq U$.

Problem 6: Let F be a field and $V = Mat_n(F)$ the vector space of all $n \times n$ matrices over F. For each $1 \leq i, j \leq n$ let $e_{ij} \in Mat_n(F)$ denote the matrix whose (i,j)-entry is equal to 1 and all other entires are equal to 0.

(a) Prove that the set $B = \{e_{ij} : 1 \leq i, j \leq n\}$ is a basis of V.

(b) Let U be the set of upper-triangular matrices in V, that is, U consists of all matrices whose (i,j)-entry is equal to 0 for all $i > j$. Prove that U is a subspace, find a basis of U and the dimension of U. **Hint:** U has a basis which is a subset of B.

Q (c) Let $W = \mathfrak{sl}_n(F)$, the set of all matrices in V with zero trace. Recall that W was shown to be a subspace in Homework#1. Let $U = Span(e_{11})$. Prove that U is a complement of W in V.

QD (d) Use (c) to find $\dim(W)$ and then find a basis of W (and prove that you indeed found a basis). **Note:** Unlike (b), W has no basis which is a subset of B, but it does have a basis where each matrix has only one or two nonzero entries.

Problem 7: Let V be a vector space over $\mathbb{C}$ with $\dim(V) = n$. Note that we can also consider V as a vector space over $\mathbb{R}$ (by restricting the scalars from $\mathbb{C}$ to $\mathbb{R}$). Prove that as an $\mathbb{R}$-vector space V has dimension $2n$. **Hint:** If $\{v_1, \ldots, v_n\}$ is a basis of V as a vector space over $\mathbb{C}$, how to get a basis of V as a vector space over $\mathbb{R}$? Use the basic example $V = \mathbb{C}$ to make a guess.

Problem 8 (bonus): (=Problem 3(c) from Homework#1, slightly modified). For every $n \in \mathbb{Z}_{\geq 0}$ construct an explicit example of a $\mathbb{Q}$-subspace W_n of $\mathbb{R}$ which has dimension n (as a vector space over $\mathbb{Q}$) and prove that the dimension is indeed n.