Homework #11. Due Thursday, December 1st, in class

Reading:

1. For this homework assignment: § 7.3, 6.8 and parts of 6.1.
2. For classes on Nov 29, Dec 1: read § 6.8 and briefly go over 6.1-6.3.

HOMEWORK POLICY: In this homework all quiz problems may be discussed with others (following the previously stated rules for QD problems).

Problem 1 Q:

(a) Prove that there exist NO matrix $A \in Mat_{3\times 3}(\mathbb{Q})$ (where $\mathbb{Q}$ denotes rationals) s.t. $A^2 = 5I$. **Hint:** Use minimal polynomials.

(b) Given an example of a matrix $A \in Mat_{3\times 3}(\mathbb{R})$ s.t. $A^2 = 5I$. Then explain where your proof from (a) would break down if $\mathbb{Q}$ is replaced by $\mathbb{R}$.

Problem 2: Q Let V be a finite-dimensional vector space over a field F, $n = \dim(V)$ and $H \in Bil(V)$. Let β be an ordered basis of V and $A = [H]_\beta$. Let $\psi : V \to F^n$ be the isomorphism given by $\psi(v) = [v]_\beta$ (where elements of F^n are thought of as column vectors).

(a) Prove that $\psi(\mathrm{RKer}(H)) = \mathrm{Ker}(A)$ (where by definition $\mathrm{Ker}(B) = \mathrm{Ker}(L_B)$ for a square matrix B).

(b) Prove that $\psi(\mathrm{LKer}(H)) = \mathrm{Ker}(A^t)$

(c) Deduce that $\dim(\mathrm{LKer}(H)) = \dim(\mathrm{RKer}(H))$ (this is Proposition 23.1)

(d) Deduce that H is non-degenerate if and only if A is invertible (this is Proposition 23.2).

Hint for (a) and (b): Recall that $H(v, w) = [v]_\beta^t A[w]_\beta$ for all $v, w \in V$. The expression on the right-hand side can be written as a dot product in two different ways – one is useful for (a) and the other one is useful for (b). Recall that we proved in class that the dot product is non-degenerate.

Definition: Let V be a vector space over a field F, and let H be a symmetric bilinear form on V. A basis β of V is called H-**orthogonal** if $H(b, b') = 0$

for any distinct elements b, b' of β. If in addition $H(b,b) = 1$ for all $b \in \beta$, then β is called **H-orthonormal**.

By Theorem 6.36 (which we will prove in class on Tue, Nov 29), for every symmetric bilinear form H there exists an H-orthogonal basis, provided that $char(F) \neq 2$.

Problem 3: Q Let F be a field, $n \in \mathbb{N}$ and $V = Mat_{n\times n}(F)$. Define the function $H : V \times V \to F$ by $H(A,B) = \mathrm{tr}(AB)$.

(a) Prove that H is a non-degenerate symmetric bilinear form on V.

(b) Assume that $char(F) \neq 2$. Find an H-orthogonal basis of V.

Hint: Recall that V has a natural basis $\{e_{ij} : 1 \le i,j \le n\}$ where e_{ij} is the matrix whose (i,j)-entry is 1 and all other entries are 0. To prove that H is non-degenerate it suffices to show that there is no nonzero $A \in Mat_{n\times n}(F)$ s.t. $tr(Ae_{ij}) = 0$ for all i, j. Note that

$$e_{ij}e_{kl} = \delta_{il}e_{jk} \text{ for all } i, j, k, l.$$

Problem 4: Q Let $A = \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix} \in Mat_{3\times 3}(\mathbb{R})$, and let H be the bilinear form on $\mathbb{R}^3$ given by $H(v,w) = v^t A w$.

(a) Use the algorithm from the proof of Theorem 6.36 to find an H-orthogonal basis of V.

(b) Use (a) to find an invertible matrix Q and a diagonal matrix D s.t. $Q^t A Q = D$.

Problem 5: The goal of this problem is to prove that if $A \in Mat_{n\times n}(\mathbb{R})$ is a symmetric matrix (that is, $A^t = A$), then

(i) A is diagonalizable (over $\mathbb{R}$)

(ii) There is an orthogonal matrix Q s.t. the matrix $Q^{-1}AQ$ is diagonal. A matrix Q is called **orthogonal** if Q is invertible and $Q^{-1} = Q^t$.

This theorem is established in § 6.3, but the proof uses many other results from § 6.1-6.3. This exercise provides a relatively short self-contained proof. Consider the function $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ given by

$$\langle (x_1, \ldots, x_n)^t, (y_1, \ldots, y_n)^t \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

where $\overline{y}$ is the complex conjugate of y. Note that $\langle \cdot, \cdot \rangle$ restricted to $\mathbb{R}^n \times \mathbb{R}^n$ is the usual dot product.

Also note that $\langle \cdot, \cdot \rangle$ is NOT a bilinear form on $\mathbb{C}^n$; it is a sesqui-linear form: it satisfies $\langle u_1 + u_2, v \rangle = \langle u_1, v \rangle + \langle u_2, v \rangle$, $\langle u, v_1 + v_2 \rangle = \langle u, v_1 \rangle + \langle u, v_2 \rangle$ and $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$; however, $\langle u, \lambda v \rangle$ is equal to $\overline{\lambda} \langle u, v \rangle$, not $\lambda \langle u, v \rangle$.

Finally, note that $\langle \cdot, \cdot \rangle$ is positive definite, that is, $\langle v, v \rangle > 0$ for all $v \neq 0$.

For every matrix $B \in Mat_{n \times n}(\mathbb{C})$ denote by B^* the matrix obtained from B by transposition followed by applying complex conjugation to every entry, that is, if $B = (b_{ij})$, then $B^* = (\overline{b_{ji}})$.

(a) Prove that $\langle Bu, w \rangle = \langle u, B^* w \rangle$ for all $u, w \in \mathbb{C}^n$ and $B \in Mat_{n \times n}(\mathbb{C})$.

(b) **Q** Now assume that $B = B^*$. Prove that every eigenvalue of B is real. **Hint:** If $\lambda \in Spec(B)$, choose $v \in E_\lambda(B)$ and apply (a) with $u = w = v$.

(c) **Q** Again assume that $B = B^*$. Prove that B is diagonalizable. **Hint:** Suppose not. Since B has JCF over $\mathbb{C}$ (as $\mathbb{C}$ is algebraically closed), it may only fail to be diagonalizable if for some $\lambda \in Spec(B)$ there is a Jordan block of size ≥ 2 in $JCF(B)$. In that case there exists $v \in \mathbb{C}^n$ s.t. $(B - \lambda I)v \neq 0$, but $(B - \lambda I)^2 v = 0$. Use (a) with B replaced by $B - \lambda I$ and suitable u and w to get a contradiction with the fact that $\langle \cdot, \cdot \rangle$ is positive definite.

In parts (d)-(g) below we assume that $A \in Mat_{n \times n}(\mathbb{R})$ and A is symmetric.

(d) Deduce from (b) and (c) that A is diagonalizable over $\mathbb{R}$.

(e) **Q** Prove that if λ and μ are distinct eigenvalues of A, then for any $u \in E_\lambda(A)$ and $w \in E_\mu(A)$ we have $\langle u, w \rangle = 0$.

(f) Use (e) to prove that $\mathbb{R}^n$ has an orthonormal (with respect to the dot product) basis consisting of eigenvectors of A.

(g) **Q** Prove that a matrix $Q \in Mat_{n \times n}(\mathbb{R})$ is orthogonal $\iff$ the columns of Q form an orthonormal basis of $\mathbb{R}^n$. Deduce that there is an orthogonal matrix Q s.t. $Q^{-1}AQ$ is diagonal. (Note that $Q^{-1}AQ = Q^t AQ$ since Q is orthogonal).

Problem 6: **Q** Let A be as in Problem 4. Find an orthogonal matrix Q and a diagonal matrix D s.t. $Q^t AQ = D$ (such Q and D exist by Problem 5). **Hint:** If you do not see how to start, read Problem 5(e),(f),(g) carefully.