Homework #10. Due Thursday, November 17th, in class

Reading:

1. For this homework assignment: § 7.2, 7.3.
2. For next week's classes: read § 6.8.

HOMEWORK POLICY: In this homework all quiz problems may be discussed with others (following the previously stated rules for QD problems).

Problem 1: Let F be a field and $A_1, \ldots, A_k$ square matrices over F (possibly of different sizes). Recall that by $A_1 \oplus \ldots \oplus A_k$ we denote the block-diagonal matrix whose diagonal blocks are $A_1, \ldots, A_k$ (in this order). Let $A = A_1 \oplus \ldots \oplus A_k$. Prove that

(a) $p(A) = p(A_1) \oplus \ldots \oplus p(A_k)$ for any polynomial $p(x) \in \mathcal{P}(F)$

(b) $\mathrm{rk}(A) = \mathrm{rk}(A_1) \oplus \ldots \oplus \mathrm{rk}(A_k)$

Problem 2: **Q** Let A be a square matrix over a field F, and assume that $\chi_A(x)$ splits. For each $\lambda \in Spec(A)$ and $n \in \mathbb{N}$ denote by $f(n, \lambda)$ the number of Jordan blocks of size n corresponding to λ in $JCF(A)$. Note that computing $JCF(A)$ is the same as computing the numbers $f(n, \lambda)$ for all $\lambda \in Spec(A)$ and $n \in \mathbb{N}$. In Lecture 20 we proved that

$$f(n, \lambda) = \mathrm{rk}((A - \lambda I)^{n-1}) - 2\mathrm{rk}((A - \lambda I)^n) + \mathrm{rk}((A - \lambda I)^{n+1}). \qquad (***)$$

In § 7.2 it is explained how to compute JCF(A) using the notion of **dot diagram**, and by Theorem 7.10(b) in the book the dot diagrams of A can be computed in terms of the quantities $\mathrm{rk}((A - \lambda I)^n)$ for $\lambda \in Spec(A)$ and $n \in \mathbb{N}$. Deduce formula (***) directly from Theorem 7.10. **Hint:** First reformulate the definition of a dot diagram in terms of sizes of Jordan blocks of A.

Problem 3: **Q** Problem 2 after § 7.2 on page 510.

Problem 4: Let F be a field and let $A \in Mat_{n \times n}(F)$ be nilpotent.

(a) Prove that $\chi_A(x)$ splits and all blocks in $JCF(A)$ correspond to 0. **Note:** This fact was established in the course of our proof of the existence of JCF, but it is also useful to see how to obtain this fact using the existence of JCF (since the latter can be proved in many different ways). One can also give an argument using minimal polynomials.

(b)**Q** Now assume in addition that $\text{rk}(A) = n-1$. Use dot diagrams to prove that $JCF(A) = J(0,n)$ – a single Jordan block of size n corresponding to 0. **Note:** The use of dot diagrams is by no means necessary, but they provide a particularly elegant argument. **Hint:** How many dots are there in the first line?

Problem 5: Q Let F be a field, $\lambda \in F$, $n \in \mathbb{N}$ and $A = J(\lambda, n)^2$. Use the above techniques to determine $JCF(A)$. **Hint:** The answer will depend on whether $\lambda \neq 0$ or not. In order to compute $J(\lambda, n)^k$ for $k \in \mathbb{N}$ it may be convenient to write $J(\lambda, n) = \lambda I + J(0, n)$ and use the binomial formula.

Problem 6: Let $A \in Mat_{n\times n}(F)$ and assume that $\chi_A(x)$ splits. Prove that A is similar to its transpose A^t. **Hint:** Use JCF.

Note: The assumption that $\chi_A(x)$ splits is not necessary, but one needs a different technique to remove it (one way to do it is using the notion of rational canonical form discussed in § 7.4).

Problem 7: Let $A \in Mat_{n\times n}(F)$ and assume that $\chi_A(x)$ splits. Suppose that $Spec(A) = \{\lambda_1, \ldots, \lambda_k\}$, and let μ_i be the maximum size of a Jordan block corresponding to λ_i in $JCF(A)$. In Lecture 21 we outlined a proof of the following formula for the minimal polynomial $\mu_A(x)$ of A:

$$\mu_A(x) = \prod_{i=1}^{k} (x - \lambda_i)^{\mu_i}.$$

The purpose of this problem is to provide the details of that proof that were left out in class:

(a) Prove that if B, C are similar matrices, then $\mu_B(x) = \mu_C(x)$.

(b) **Q** Suppose that $A = \oplus_{i=1}^k A_i$. Prove that $\mu_A(x) = LCM(\mu_{A_1}(x), \ldots, \mu_{A_k}(x))$ where by definition $LCM(f_1(x), \ldots, f_k(x))$ is the monic polynomial of the smallest possible degree which is divisible by each $f_i(x)$.

Problem 8: Let F be an algebraically closed field.

(a) Let A, B be 2×2 matrices over F, and suppose that $\mu_A(x) = \mu_B(x)$ and $\chi_A(x) = \chi_B(x)$, that is, A and B have the same minimal polynomial and the same characteristic polynomial. Prove that A and B are similar. **Hint:** Show that $JCF(A)$ is completely determined by $\chi_A(x)$ and $\mu_A(x)$ (when A is 2×2).

(b) **Q** Prove that the assertion of (a) remains true for $n = 3$.

(c) **Q** Give an example showing that the assertion of (a) may be false for $n = 4$.